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Determination of the tangents for a real plane algebraic curve

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Abstract

The purpose of this paper is to present an algorithm for computing the tangents of a real plane algebraic curve. By this algorithm, all the slopes of the tangents to a real plane algebraic curve at a particular point may be accurately represented via polynomial real root isolation.

Keywords: Real plane algebraic curve; Tangent; Tangential ray; Slope; Signed subresultant sequence; Transfer principle

0. Introduction

As a geometric feature of real curves, the concept of tangents plays an important role in the study of real plane algebraic curves. The tangents to a real plane curve at one of its points reflect the status of this curve passing through the point. Let $\mathcal{C}$ be a real plane algebraic curve in the real plane $\mathbb{R}^2$, P a particular point of $\mathcal{C}$. By Proposition 9.5.1 in Bochnak et al. (1998), for a sufficiently small open disk U with centre P, $\mathcal{C} \cap (U \setminus \{P\})$ consists of a finite number of semi-algebraic connected components $\mathcal{B}_1, \ldots, \mathcal{B}_k$ and there is a semi-algebraic homeomorphism π_i of the half open-closed interval $[0, 1[$ to B_i such that $\pi_i(0) = P$ for every $i = 1, \ldots, n$. By abusing Definition 9.5.2 in Bochnak et al. (1998), these components $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are called the *half-branches* of $\mathcal{C}$ centred at P in this paper. The number k of the components $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is independent of the radius of U, and $k = 0$ if and only if P is isolated.

According to the classical definition of tangents in real geometry, the concepts related to the tangents of a real plane algebraic curve may be stated as follows:

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Definition. Let $\mathcal{C}$ be as above, let P be a particular point of $\mathcal{C}$ with coordinates (a, b), and $\mathcal{B}$ a half-branch of $\mathcal{C}$ centred at P. If there exists an $\lambda \in \mathbb{R} \cup \{\infty, -\infty\}$ such that $\lim_{Q \to P} \frac{y-b}{x-a} = \lambda$ for $Q \in \mathcal{B}$ with coordinates (x, y) different from (a, b), the ray with initial point P, which is the limit position of the ray $\overrightarrow{PQ}$ as Q approaches P along the half-branch $\mathcal{B}$, is called the *tangential ray* to $\mathcal{B}$ at P, and λ is called the *slope* of the tangential ray to $\mathcal{B}$ at P.

In this case, we shall say that the line $\lambda(x - a) - (y - b) = 0$ or $x - a = 0$ is the *tangent* to $\mathcal{B}$ at P according as $\lambda \in \mathbb{R}$ or $\lambda \in \{\infty, -\infty\}$. The tangent (tangential ray) to $\mathcal{B}$ at P is called a *tangent* (*tangential ray*) to $\mathcal{C}$ at P.

It is a classical fact that every half-branch centred at P has a unique tangential ray at P. For the details, also refer to Theorem 2.4 in Section 2.

Clearly, a tangential ray to $\mathcal{B}$ at P is a half of the associated tangent. Thereby, two tangential rays to $\mathcal{C}$ at P are perhaps distinct, even if their associated tangents are the same. For example, let $\mathcal{C}$ be the circle defined by $x^2 + y^2 = 1$, P the point of $\mathcal{C}$ with coordinates $(1, 0)$. Obviously, $\mathcal{C}$ has exactly two half-branches centred at P: one upward, another downward. It is easy to see that the two half-branches possess the upward and downward vertical tangential rays at P with slopes $-\infty$ and ∞ respectively. But the line $x - 1 = 0$ is the only tangent to $\mathcal{C}$ at P.

Let $\mathcal{C}$ be a curve in the real plane $\mathbb{R}^2$ defined by the polynomial equation $f(x, y) = 0$, and P a particular point of $\mathcal{C}$ with coordinates (a, b). If P is not singular, i.e. not all of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ vanish at P, then the line $\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) = 0$ is the only tangent to $\mathcal{C}$ at P. When P is singular, the following method of computing the tangents to $\mathcal{C}$ at P is presented in some literature (see, for example, Walker (1978) or section 4.2 in Sakkalis and Farouki (1990)):

> If m is the minimal natural number such that not all derivatives of order m of $f(x, y)$ vanish at P, then the ratios $\lambda : \mu$ satisfying the following equality:
>
> $$\frac{\partial^m f}{\partial x^m}\lambda^m + \binom{m}{1}\frac{\partial^m f}{\partial x^{m-1}\partial y}\lambda^{m-1}\mu + \cdots + \binom{m}{m}\frac{\partial^m f}{\partial y^m}\mu^m = 0 \qquad (\star)$$
>
> correspond exactly to the tangents $\mu(x - a) - \lambda(y - b) = 0$ at P.

However, in the argument about the above result, the ground field is usually assumed to be algebraically closed. Since the curves considered in this paper are real, our definition of tangents should be classical. Thereby, when the above method is adopted, the following questions arise naturally:

(1) Must every such real ratio $\lambda : \mu$ as above correspond to a tangent to $\mathcal{C}$ at P?

(2) Which ratio corresponds to a tangent to $\mathcal{C}$ at P if the answer to question (1) is No?

Actually, the answer to question (1) is No; see the example as follows:

Example. Consider the real curve $\mathcal{C}$: $2x^5 - x^4y + xy^2 - y^3 = 0$. It is easy to see that the origin is a singular point of $\mathcal{C}$. In this case, the corresponding equality $(\star)$ is $6\lambda\mu^2 - 6\mu^3 = 0$. Thereby, we get two real ratios 1:1, 1:0 satisfying the equality.

However, as is proved in what follows, the line $x - y = 0$ is the only tangent to $\mathcal{C}$ at the origin; in other words, there is no real tangent, which corresponds to the real ratio 1:0, to $\mathcal{C}$ at the origin.

The determination of tangents is useful in tracing a real algebraic curve through a singular point. As is discussed by Bajaj et al. (1988) and Hoffman (1988), this determination has practical applications in geometric modelling. In this paper, we will present an effective method for computing the tangents of a real plane algebraic curve. By this method, for a point P of an algebraic curve $\mathcal{C}$ in $\mathbb{R}^2$, we may compute accurately the slopes of all tangential rays to $\mathcal{C}$ at

P via polynomial real root isolation, and determine all the half-branches possessing the same tangential ray at P. The main technique of this paper is to use signed subresultant sequences (i.e., Sturm–Habicht sequences) in counting real roots of univariate polynomials.

1. Preliminaries

Before establishing the main results, we need some preliminaries. First, we extend the field $\mathbb{R}$ of real numbers to an ordered field containing an infinitesimal element ϵ.

Let ϵ be an indeterminate over $\mathbb{R}$. Then the ordering $\leq$ of $\mathbb{R}$ can be extended uniquely to an ordering of the field $\mathbb{R}(\epsilon)$, denoted still by $\leq$, such that ϵ is a positive and infinitesimal element over $\mathbb{R}$. Obviously, for a non-zero element $\frac{g}{h} \in \mathbb{R}(\epsilon)$ with $g, h \in \mathbb{R}[\epsilon]$, $\frac{g}{h} < 0$, if and only if the trailing coefficient of gh is negative as a univariate polynomial in ϵ over $\mathbb{R}$. Thereby, for every non-zero $g \in \mathbb{R}[\epsilon]$, we have $\mathrm{sign}(g) = \mathrm{sign}(\mathrm{tcoeff}(g; \epsilon))$, where $\mathrm{tcoeff}(g; \epsilon)$ stands for the trailing coefficient of g as a univariate polynomial in ϵ over $\mathbb{R}$, and $\mathrm{sign}(g)$, $\mathrm{sign}(\mathrm{tcoeff}(g; \epsilon))$ are the signs of g, $\mathrm{tcoeff}(g; \epsilon)$ with respect to the orderings of $\mathbb{R}(\epsilon)$, $\mathbb{R}$ respectively.

Denote by R the real closure of $(\mathbb{R}(\epsilon), \leq)$. Of course, assume that $\mathbb{R} \subset R$. Moreover, construct the two subsets of R as follows:

$$A := \{z \in R \mid \text{For some positive number } d \in \mathbb{R}, -d \leq z \leq d\},$$
$$M := \{z \in R \mid \text{For every positive number } d \in \mathbb{R}, -d \leq z \leq d\}.$$

Obviously, M consists of all elements in R infinitesimal over $\mathbb{R}$. By the structure of the ordering $\leq$, we have $\mathbb{R}[\epsilon] \subset A$, and $\epsilon \in M$. By the familiar facts on real valuations (see Proposition 1.3 in Knebusch (1973) or the relevant theorems in section 5 of Lam (1980)), A is a real valuation ring of R, and M is the only maximal ideal of A. Moreover, (A, M) is compatible with $\leq$; in other words, both A and M are convex in R with respect to $\leq$. In what follows, every element in A is called bounded (over $\mathbb{R}$), but every element in $R \setminus A$ is called unbounded (over $\mathbb{R}$).

Let π be the real place associated with the valuation ring A. Then π is a mapping of R into $\mathbb{R} \cup \{\infty\ -\infty\}$ satisfying the following conditions:

(1.1) The restricted mapping $\pi\ |_A$ is an $\mathbb{R}$-homomorphism of A onto $\mathbb{R}$ such that M is exactly the kernel of $\pi\ |_A$.

(1.2) For ξ in $R \setminus A$, $\pi(\xi) = \infty$ if ξ is positive, otherwise $\pi(\xi) = -\infty$.

(1.3) For any $\alpha, \beta \in R$, $\alpha \leq \beta$ implies $\pi(\alpha) \leq \pi(\beta)$, where we adopt the convention: $-\infty < r < \infty$ for all $r \in \mathbb{R}$.

By condition (1.1), we have $\pi\big(g(\epsilon)\big) = g(0)$ for every $g \in \mathbb{R}[z]$, and $\alpha - \pi(\alpha) \in M$ for all $\alpha \in A$.

In what follows, we adopt the usual symbols as follows: $]a, b[$ (or $[a, b]$) stands for the open (or closed) interval in $\mathbb{R}$ with endpoints a, b, and but $]a, b[_R$ (or $[a, b]_R$) stands for the open (or closed) interval in R with endpoints a, b.

Now let $\theta \in M$ be a positive element. Then θ is infinitesimal over $\mathbb{R}$. Clearly, θ is transcendental over $\mathbb{R}$. Thereby, for an indeterminate z, every element in $\mathbb{R}[\theta][z]$ may be considered as a polynomial in θ over $\mathbb{R}[z]$. For a non-zero $g(z) \in \mathbb{R}[\theta][z]$, denote by $\mathrm{tcoeff}(g(z); \theta)$ the trailing coefficient of $g(z)$ as a polynomial in θ over $\mathbb{R}[z]$. Clearly, $\mathrm{tcoeff}(g(z); \theta) \in \mathbb{R}[z]$.

Lemma 1.1. *Let θ and $g(z)$ be as above. If $B \in \mathbb{R}$ is an upper bound for every real root of tcoeff$(g(z); \theta)$, i.e., $|e| < B$ for every real root e of tcoeff$(g(z); \theta)$, then every bounded root of $g(z)$ in R belongs to $]-B, B[_R$.*

Proof. Suppose that the lemma above is false. Then there is at least one bounded root α of $g(z)$ in R such that either $\alpha \leq -B$ or $\alpha \geq B$. By the hypothesis, put $g(z) = \theta^s g_0(\theta, z)$, where $s \geq 0$, and $g_0(\theta, z)$ is a polynomial in θ over $\mathbb{R}[z]$ such that $g_0(0, z) \neq 0$. Clearly, $\operatorname{tcoeff}(g(z); \theta) = g_0(0, z)$, and $g_0(\theta, \alpha) = 0$. Observe that $\theta \in M$ and $\alpha \in A$. Then $\pi(\theta) = 0$, and $\pi(\alpha) \in \mathbb{R}$. So we have $\pi\big(g_0(\theta, \alpha)\big) = 0$, i.e., $g_0(0, \pi(\alpha)) = 0$. This implies that $\pi(\alpha)$ is a real root of $\operatorname{tcoeff}(g(z); \theta)$. By condition (1.3), we have either $\pi(\alpha) \leq \pi(-B) = -B$ or $\pi(\alpha) \geq \pi(B) = B$; this contradicts the hypothesis of Lemma 1.1. The proof is completed. □

For any polynomial g in $\mathbb{R}[x, y]$, write $\operatorname{lcoeff}(g; y)$, $\deg(g; y)$ respectively for the leading coefficient and the degree of g as a univariate polynomial in y over $\mathbb{R}[x]$. Then $\operatorname{lcoeff}(g; y) \in \mathbb{R}[x]$, and $\deg(g; y)$ is a non-negative integer if $g \neq 0$. Now, let f be a polynomial in $\mathbb{R}[x, y]$ such that $\frac{\partial f}{\partial y} \neq 0$. According to the definition of signed subresultants (see Notation 8.52 in Basu et al. (2003) or Definition 2 in Gonzalez-Vega et al. (1998)), we may obtain the following sequence of polynomials in $\mathbb{R}[x, y]$:

$$f_d, f_{d-1}, \ldots, f_0,$$

where $d = \deg(f; y)$, and f_i is the i-th signed subresultant of f and $\frac{\partial f}{\partial y}$ as polynomials over $\mathbb{R}[x]$ in one variable y, $i = 0, \ldots, d$.

In what follows, the sequence $f_d, f_{d-1}, \ldots, f_0$ is called *the signed subresultant sequence of $f(x, y)$ relative to y*.

Proposition 1.2. *Let $g(z)$ be as in Lemma 1.1. Then we may effectively compute the numbers of bounded and unbounded roots of $g(z)$ in R.*

Proof. According to Lemma 1.1, we implement the following effective computations:

Step 1. Compute the signed subresultant sequence of $g(z)$ relative to z, and delete those polynomials that are identically 0 from this sequence. Then, a sequence of polynomials is obtained as follows:

$$g_m = g(z), g_{m-1}, \ldots, g_0,$$

where $m \leq \deg(g; z)$, and $g_i \in \mathbb{R}[\theta][z]$, $i = 0, \ldots, m$.

Step 2. Extract the trailing coefficient $u_i(z)$ of g_i as a polynomial in θ and the leading coefficient $v_i(\theta)$ of g_i as a polynomial in z, $i = 0, \ldots, m$. Note: $u_i(z) \in \mathbb{R}[z]$ but $v_i(\theta) \in \mathbb{R}[\theta]$ for all i.

Step 3. Extract the leading coefficient a_i of $u_i(z)$ and the trailing coefficient b_i of $v_i(\theta)$, $i = 0, \ldots, m$.

Step 4. Compute respectively the modified numbers W_1, W_2, W_3 and W_4 of sign changes in the lists $\langle (-1)^{\deg(u_i;z)} a_i \mid i = 0, \ldots, m \rangle$, $\langle a_i \mid i = 0, \ldots, m \rangle$, $\langle (-1)^{\deg(g_i;z)} b_i \mid i = 0, \ldots, m \rangle$ and $\langle b_i \mid i = 0, \ldots, m \rangle$.

Then, we have the following claims:

(1) The number of bounded roots of $g(z)$ in R is $W_1 - W_2$.

(2) The number of unbounded roots of $g(z)$ in R is $W_2 + W_3 - W_1 - W_4$.

Indeed, by Corollary 9.33 in Basu et al. (2003), it is clear that the number of roots of $g(z)$ in R is $V_1 - V_2$, where V_1, V_2 are the modified numbers of sign changes in the lists $\langle (-1)^{\deg(g_i;x)} v_i(\theta) \mid i = 0, \ldots, m \rangle$ and $\langle v_i(\theta) \mid i = 0, \ldots, m \rangle$. Since θ is positive and infinitesimal over $\mathbb{R}$, we have $\operatorname{sign}(v_i(\theta)) = \operatorname{sign}(b_i)$ for $i = 0, \ldots, m$. So we have $V_1 - V_2 = W_3 - W_4$.

Take arbitrarily an upper bound B_0 for every real root of $\operatorname{tcoeff}(g(z); \theta)$. Obviously, there is a sufficiently large element $B \in \mathbb{R}$ such that $B_0 < B$, and $\operatorname{sign}(u_i(B)) = \operatorname{sign}(a_i)$ for

$i = 0, \ldots, m$. Observe that the trailing coefficients of $g_i(\theta, -B)$, $g_i(\theta, B)$ as polynomials in θ are $(-1)^{\deg(u_i;x)} u_i(-B)$, $u_i(B)$ respectively. In this case, the modified numbers of sign changes in the lists $\langle g_i(\theta, -B) \mid i = 0, \ldots, m\rangle$, $\langle g_i(\theta, B) \mid i = 0, \ldots, m\rangle$ are W_1, W_2 respectively. By Corollary 9.33 in Basu et al. (2003), the number of roots of $g(z)$ in the open interval $]-B, B[$ is $W_1 - W_2$. According to Lemma 1.1, the number of bounded roots of $g(z)$ in R is $W_1 - W_2$. This completes the proof. □

Remark. By the proof of Proposition 1.2, it is easy to see that the numbers of negative and positive unbounded roots of $g(z)$ in R are $W_3 - W_1$, $W_2 - W_4$ respectively.

By an argument similar to the proof of Lemma 1.1, we may further establish the following lemma.

Lemma 1.3. *Let the notations be as in Lemma 1.1. If $]c_1, d_1[, \ldots,]c_s, d_s[$ are disjoint open intervals in $\mathbb{R}$ such that $e \in \bigcup_{1\le k\le s}]a_k, b_k[$ for every real root e of tcoeff$(g(z); \theta)$, then every bounded root of $g(z)$ in R belongs to $\bigcup_{1\le k\le s}]a_k, b_k[_R$.*

Definition. Let $h(z)$ be a univariate polynomial in $\mathbb{R}[z]$. A sequence of open intervals $]c_1, d_1[, \ldots,]c_t, d_t[$ in $\mathbb{R}$ is called a set of isolating intervals for $h(z)$, if the following conditions are satisfied:

(1) $-\infty < c_1 < d_1 \le c_2 < d_2 \le \cdots \le c_t < d_t < \infty$.

(2) For every $k \in \{1, \ldots, t\}$, there is exactly one root of $h(z)$ in $]c_k, d_k[$.

(3) Every root of $h(z)$ in $\mathbb{R}$ belongs to $\bigcup_{1\le k\le t}]c_k, d_k[$.

By Lemma 1.3, we may prove the following.

Theorem 1.4. *Let $g(z)$ be as in Lemma 1.1. Then we may effectively compute a univariate polynomial $h(z) \in \mathbb{R}[z]$ and a finite number of open intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ in $\mathbb{R}$ such that the following statements are true:*

(1) $-\infty < c_1 < d_1 \le c_2 < d_2 \le \cdots \le c_t < d_t < \infty$.

(2) *For every $k \in \{1, \ldots, s\}$, there is exactly one root of $h(z)$ in $]c_k, d_k[$.*

(3) *For every $k \in \{1, \ldots, s\}$, there is at least one root of $g(z)$ in $]c_k, d_k[_R$.*

(4) *Every bounded root of $g(z)$ in R belongs to $\bigcup_{1\le k\le s}]c_k, d_k[_R$.*

(5) *If α is a root of $g(z)$ in $]c_\ell, d_\ell[_R$ for some $\ell \in \{1, \ldots, s\}$, then $\pi(\alpha)$ is the root of $h(z)$ in $]c_\ell, d_\ell[$.*

Proof. According to Lemma 1.3, we implement the following effective computations:

Step 1. Take $h(z)$ as the trailing coefficient tcoeff$(g(z); \theta)$ of $g(z)$ as a polynomial in the variable θ. Note: $h(z) \in \mathbb{R}[z]$.

Step 2. By real root isolation for polynomials (see Algorithm 10.41 in Basu et al. (2003)), find out a set of isolating intervals $]c_1, d_1[, \ldots,]c_t, d_t[$ for $h(z)$.

Step 3. For every $k \in \{1, \ldots, t\}$, by Corollary 9.33 in Basu et al. (2003), check whether $g(z)$ has a root in the open interval $]c_k, d_k[_R$. Then collect all indexes k such that $g(z)$ has a root in $]c_k, d_k[_R$.

Then, we may assert that the polynomial $h(z)$ and the intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ are as required in the proposition whenever $\{1, \ldots, s\}$ is the set of all collected indexes.

Indeed, statements (1–3) in the proposition are obviously true. By Lemma 1.3, it is easy to see that statement (4) is true. Now assume that α is a root of $g(z)$ in $]c_\ell, d_\ell[_R$ for some $\ell \in \{1, \ldots, s\}$. By the structure of the valuation ring A, $\alpha \in A$. Put $g(z) = \theta^s g_0(\theta, z)$, where $s \ge 0$, and $g_0(\theta, z)$ is a polynomial in θ over $\mathbb{R}[z]$ such that $g_0(0, z) \ne 0$. Clearly, $h(z) = \text{tcoeff}(g(z); \theta) = g_0(0, z)$, and $g_0(\theta, \alpha) = 0$. So we have

$$g_0\big(0, \pi(\alpha)\big) = \pi(g_0(\theta, \alpha)) = 0.$$

Thereby, $\pi(\alpha)$ is a root of $h(z)$ in $\mathbb{R}$. From the inequalities $c_\ell < \alpha < d_\ell$, we get

$$c_\ell = \pi(c_\ell) \le \pi(\alpha) \le \pi(d_\ell) = d_\ell.$$

This implies that $\pi(\alpha)$ must be the only root of $h(z)$ in $]c_\ell, d_\ell[$, because the open interval $]c_\ell, d_\ell[$ contains exactly one real root of $h(z)$. Therefore, our assertion is verified. □

2. Determination of tangents

In this section, we will present an algorithm computing the tangents to a real plane algebraic curve at a particular point.

In this section, F will denote a computable ordered subfield of $\mathbb{R}$. For example, $F = \mathbb{Q}$ or $F = \mathbb{Q}(\beta)$, where $\mathbb{Q}$ is the field of rational numbers, and β is a real algebraic number. Note: Algorithms for arithmetic in $\mathbb{Q}(\beta)$ can be found in Loos (1982). Throughout this section, the following symbols will be kept: $\mathcal{C}$ is a curve in $\mathbb{R}^2$ defined by the equation $f(x, y) = 0$, and P is a particular point of $\mathcal{C}$ with coordinates (a, b), where $f(x, y) \in F[x, y]$, and $a, b \in F$ such that $f(a, b) = 0$. Without loss of generality, we may assume that $f(x, y)$ is primitive as a polynomial over $F[x]$ in the variable y, i.e., all the coefficients of $f(x, y)$ as a polynomial over $F[x]$ in the variable y have no non-constant common factor. The primitivity of $f(x, y)$ assures that the number of intersection points of every vertical line with $\mathcal{C}$ is finite.

According to the geometric properties of real plane algebraic curves (cf. the relevant facts in section 2 of Cucker et al. (1991) or in section 3 of Hong (1996)), there exist two positive numbers δ, Δ such that the following conditions hold:

(2.1) In the box $]a - \delta, a + \delta[\times]b - \Delta, b + \Delta[$, the portion of $\mathcal{C}$ consists exactly of half-branches centred at the point P.

(2.2) $\mathcal{C}$ does not intersect the top and bottom sides of the box $]a - \delta, a + \delta[\times]b - \Delta, b + \Delta[$.

(2.3) When r varies in $]a, a + \delta[$, the number of intersection points of $\mathcal{C}$ with the vertical lines $x = r$ is constant, and the number of intersection points of $\mathcal{C}$ with the vertical lines $x = r$ also is constant when r varies in $]a - \delta, a[$.

(2.4) The univariate polynomial $f(a, y)$ has the only root b in the closed interval $[b - \Delta, b + \Delta]$.

A half-branch of $\mathcal{C}$ centred at P in the box $]a, a + \delta[\times]b - \Delta, b + \Delta[$ is called a *right-half-branch* centred at P (for short, a *right-branch* centred at P), and a half-branch of $\mathcal{C}$ centred at P in the box $]a - \delta, a[\times]b - \Delta, b + \Delta[$ is called a *left-half-branch* centred at P (for short, a *left-branch* centred at P).

Now, for such a curve as above with point $P(a, b)$, define two new polynomials over $\mathbb{R}[\epsilon]$ in the variable z as follows:

$$\phi_+(\epsilon, z) := f(a + \epsilon, b + z), \quad \text{and} \quad \phi_-(\epsilon, z) := f(a - \epsilon, b - z).$$

Lemma 2.1. *Let the notations be as above. Then, for two natural numbers s and t, the following statements are equivalent:*

(1) *s and t are the numbers of right-branches and left-branches centred at P respectively.*

(2) *s and t are the numbers of roots of $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ in $]-\Delta, \Delta[_R$ respectively.*

(3) *s and t are the numbers of roots of $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ in M respectively.*

(4) *s and t are the numbers of roots of $f(a + \delta, y)$ and $f(a - \delta, y)$ in $]b - \Delta, b + \Delta[$ respectively.*

Proof. The equivalence of statements (1) and (4) follows immediately from conditions (2.1), (2.2) and (2.3).

Let s and t be the numbers of right-branches and left-branches centred at P respectively. By conditions (2.1), (2.2) and (2.3), the following sentence is valid in $\mathbb{R}$:

$$\forall x \left(a < x < a+\delta \to \exists (z_1, \ldots, z_s) \left(\left(\bigwedge_{1 \le i < j \le s} z_i \neq z_j \right) \wedge \left(\bigwedge_{1 \le i \le s} f(x, b+z_i) = 0 \right) \wedge \right.\right.$$
$$\left.\left. \left(\bigwedge_{1 \le i \le s} -\Delta < z_i < \Delta \right) \wedge \forall z \left(f(x, b+z) = 0 \wedge -\Delta < z < \Delta \to \bigvee_{1 \le i \le s} z = z_i \right) \right) \right).$$

By the familiar Transfer principle for real closed fields (see Theorem 2.78 in Basu et al. (2003) or Proposition 5.2.3 in Bochnak et al. (1998)), this sentence is also valid in R. Observe that $a < a+t < a+\delta$. This sentence implies that the number of roots of $\phi_+(\epsilon, z)$ in $]-\Delta, \Delta[_R$ is just s. Likewise, it may be verified that the number of roots of $\phi_-(\epsilon, z)$ in $]-\Delta, \Delta[_R$ is just t.

Now let α be any root of $\phi_+(\epsilon, z)$ in $]-\Delta, \Delta[_R$. Then we have

$$f\big(a, b+\pi(\alpha)\big) = \pi\big(f(a+\epsilon, b+\alpha)\big) = \pi\big(\phi_+(\alpha)\big) = \pi(0) = 0.$$

From $-\Delta < \alpha < \Delta$, it follows that $-\Delta \le \pi(\alpha) < \Delta$. By condition (2.4), necessarily $\pi(\alpha) = 0$, i.e., $\alpha \in M$. Observe $M \subseteq]-\Delta, \Delta[_R$. Hence, the number of roots of $\phi_+(\epsilon, z)$ in M is also s. Similarly, it may be verified that the number of roots of $\phi_-(\epsilon, z)$ in M is t. The proof is completed. □

Proposition 2.2. *Let the notations be as above. Then we can effectively compute the numbers of right-branches and left-branches of $\mathcal{C}$ centred at P.*

Proof. Put $\Psi_+(\epsilon, z) := z^d f(a+\epsilon, b+z^{-1})$ and $\Psi_-(\epsilon, z) := z^d f(a-\epsilon, b+z^{-1})$, where d is the degree of $f(x, y)$ as a polynomial in y. Obviously, $\Psi_+(\epsilon, z), \Psi_-(\epsilon, z) \in F[\epsilon][z]$. According to the remark after Proposition 1.2, we may effectively compute the numbers s, t of unbounded roots of $\Psi_+(\epsilon, z)$ and $\Psi_-(\epsilon, z)$. Obviously, for $\alpha \in R$, α is an unbounded root of $\Psi_+(\epsilon, z)$ (or $\Psi_-(\epsilon, z)$) if and only if α is a non-zero root of $\phi_+(\epsilon, z)$ (or $\phi_-(\epsilon, z)$) in M. Thereby, the number of roots of $\phi_+(\epsilon, z)$ in M is $s+1$ or s according as $\phi_+(\epsilon, 0) = 0$ or $\phi_+(\epsilon, 0) \neq 0$. Likewise, the number of roots of $\phi_-(\epsilon, z)$ in M may be effectively determined. According to Lemma 2.1, the numbers of right-branches and left-branches of $\mathcal{C}$ centred at P may be effectively determined. This completes the proof. □

In Arnon et al. (1984), an alternative approach is provided by the box adjacency algorithm.

Lemma 2.3. *Let the notations be as above. Then we have:*

(1) *If $\phi_+(\epsilon, \eta) = 0$ for some $\eta \in M$, then $\pi(\frac{\eta}{\epsilon})$ is the slope of the tangential ray to some right-branch of $\mathcal{C}$ at P.*

(2) *If $\phi_-(\epsilon, \eta) = 0$ for some $\eta \in M$, then $\pi(\frac{\eta}{\epsilon})$ is the slope of the tangential ray to some left-branch of $\mathcal{C}$ at P.*

Proof. It is sufficient to prove statement (1); statement (2) may be similarly proved. Let $\mathcal{R}_1, \ldots, \mathcal{R}_s$ be all distinct right-branches of $\mathcal{C}$ centred at P arranged in the order from below to above. By Lemma 2.1, we may assume that $\eta_1, \ldots, \eta_s$ are all roots of $\phi_+(\epsilon, z) = 0$ in M such that $\eta_1 < \cdots < \eta_s$. Without loss of generality, we further assume that $\eta = \eta_1$.

Now, we proceed to prove that $\pi(\frac{\eta}{\epsilon})$ is the slope of the tangential ray to $\mathcal{R}_1$ at P by considering the following possible cases:

Case 1. $\pi(\frac{\eta}{\epsilon}) = \infty$, i.e., $\frac{\eta}{\epsilon} \in R \setminus A$ is a positive unbounded element. Assume that $x_0, z_1, \ldots, z_s \in R$ such that $0 < x_0 - a < \epsilon$, $z_1 < \cdots < z_s$, and $f(x_0, z_1) = \cdots = f(x_0, z_s) = 0$. Put $\theta := x_0 - a$. Then $\theta > 0$ and $\theta \in M$, i.e., θ is positive and infinitesimal over $\mathbb{R}$. Thereby, when both $\mathbb{R}(\epsilon)$ and $\mathbb{R}(\theta)$ are regarded as ordered subfields of R, there is an order-preserving $\mathbb{R}$-isomorphism σ such that $\sigma(\epsilon) = \theta$. Obviously, R is the real closure of both $\mathbb{R}(\epsilon)$ and $\mathbb{R}(\theta)$. By Lemma 3.8 in Prestel (1984) and Zorn's lemma, it may be proved that σ can be extended to an order-preserving automorphism of R, denoted still by σ. So, for $i = 1, \ldots, s$, we have

$$f(x_0, b + \sigma(\eta_i)) = \sigma(f(a + \epsilon, b + \eta_i)) = \sigma(\phi_+(\epsilon, \eta_i)) = \sigma(0) = 0.$$

Observe that $\sigma(\eta_1) < \cdots < \sigma(\eta_s)$, and all the roots of $f(x_0, b + z)$ must be $z_1 < \cdots < z_s$. Necessarily, $\sigma(\eta_1) = z_1$, i.e., $\sigma(\eta) = z_1$. Now let D be any positive number D. Since $\frac{\eta}{\epsilon}$ is a positive unbounded element in R, we have $\frac{\eta}{\epsilon} > D$. So we have $\sigma(\frac{\eta}{\epsilon}) > D$, and $\frac{z_1}{x_0 - a} > D$. By viewing u as ϵ, the following sentence is actually verified to be valid in R for any positive number D:

$$\exists u \left(0 < u < \delta \wedge \forall (x, z_1, \ldots, z_s) \left(0 < x - a < u \wedge z_1 < \cdots < z_s \wedge \right.\right.$$
$$\left.\left. f(x, b + z_1) = \cdots = f(x, b + z_s) = 0 \to \frac{z_1}{x - a} > D\right)\right).$$

Observe that all the constants in the sentence above belong to $\mathbb{R}$. By the Transfer Principle, this sentence is also valid in $\mathbb{R}$. Hence, for any positive number D, there is $u \in \mathbb{R}$ such that the following sentence is valid in $\mathbb{R}$:

$$0 < u < \delta \wedge \forall (x, z_1, \ldots, z_s) \left(0 < x - a < u \wedge z_1 < \cdots < z_s \wedge \right.$$
$$\left. f(x, b + z_1) = \cdots = f(x, b + z_s) = 0 \to \frac{z_1}{x - a} > D\right).$$

Let x_0 be any real number such that $0 < x_0 - a < u$. Then $x_0 \in]a, a + u[\subset]a, a + \delta[$, and the line $x = x_0$ intersects the half-branch $\mathcal{C}_i$ at the only point Q_i, $i = 1, \ldots, s$. Denote by (x_0, y_i) the coordinates of Q_i, and put $z_i = y_i - b$, $i = 1, \ldots, s$. Clearly, $z_1 < \cdots < z_s$, and $f(x_0, b + z_1) = \cdots = f(x_0, b + z_s) = 0$. By the validity of the second sentence, we have $\frac{z_1}{x_0 - a} > D$, i.e., $\frac{y_1 - b}{x_0 - a} > D$. According to the definition of limits in analysis, we have $\lim_{x_0 \to a+0} \frac{y_1 - b}{x_0 - a} = \infty$. Observe that $\frac{y_1 - b}{x_0 - a}$ is just the slope of the secant $\overline{PQ_1}$ of $\mathcal{R}_1$ passing through P, Q. By the definition of tangential rays in the Introduction, the slope of the tangential ray to $\mathcal{R}_1$ at P is ∞.

Case 2. $\pi(\frac{\eta}{\epsilon}) = -\infty$, i.e., $\frac{\eta}{\epsilon} \in R \setminus A$ is a negative unbounded element. In this case, it may be similarly proved that $-\infty$ is the slope of the tangential ray to $\mathcal{R}_1$ at P.

Case 3. $\pi(\frac{\eta}{\epsilon}) = r \in \mathbb{R}$. In this case, by an argument similar to that of Case 1, we may prove that the following sentence is valid in R for any positive number d:

$$\exists u \left(0 < u < \delta \wedge \forall (x, z_1, \ldots, z_s) \left(0 < x - a < u \wedge z_1 < \cdots < z_s \wedge \right.\right.$$
$$\left.\left. f(x, b + z_1) = \cdots = f(x, b + z_s) = 0 \to -d < \frac{z_1}{x - a} - r < d\right)\right).$$

Likewise, based on the validity of the sentence above, it may be verified that the slope of the tangential ray to $\mathcal{R}_1$ at P is r. Hence, the proof is completed. □

Actually, by a copy of the proof of Lemma 2.3, we may establish the following further result:

Theorem 2.4. *Let the notations be as above, then we have:*

(1) If $\mathcal{R}_1, \ldots, \mathcal{R}_s$ are all distinct right-branches of $\mathcal{C}$ centred at P arranged in the order from below to above, and all roots of $\phi_+(\epsilon, z)$ in M are $\eta_1 < \cdots < \eta_s$, then $\pi(\frac{\eta_i}{\epsilon})$ is the slope of the tangential ray to $\mathcal{R}_i$ at P, $i = 1, \ldots, s$.

(2) If $\mathcal{L}_1, \ldots, \mathcal{L}_t$ are all distinct left-branches of $\mathcal{C}$ centred at P arranged in the order from above to below, and all roots of $\phi_-(\epsilon, z)$ in M are $\eta_1 < \cdots < \eta_t$, then $\pi(\frac{\eta_j}{\epsilon})$ is the slope of the tangential ray to $\mathcal{L}_j$ at P, $j = 1, \ldots, t$.

Theorem 2.4 reveals such a basic fact that every half-branch centred at P has a unique tangential ray at P.

In what follows, for a univariate polynomial $h(z) \in \mathbb{R}[z]$ and an open interval $]a, b[$ in $\mathbb{R}$ such that $h(z)$ has exactly one root in $]a, b[$, the only real root of $h(z)$ in $]a, b[$ is represented by $(h(z); a, b)$. For a tangential ray to $\mathcal{C}$ at P, its slope is called finite if the slope is a real number; otherwise, the slope is called infinite.

Lemma 2.5. *Let the notations be as above. Then we can effectively compute all finite slopes of tangential rays to $\mathcal{C}$ at P.*

Proof. It suffices to compute all finite slopes of tangential rays at P for right-branches centred at P; the computation for left-branches centred at P may be similarly implemented.

Based on the idea of blowing-up, the following polynomial $g(z)$ over $F[\epsilon]$ in one variable z is considered:

$$g(z) := f(a + \epsilon, b + \epsilon z).$$

According to Theorem 1.4 and its proof, we may effectively compute a polynomial $h(z) \in F[z]$ and a finite number of open intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ in $\mathbb{R}$ such that $h(z) = \mathrm{tcoeff}(g(z); \epsilon)$, and statements (1–5) in Theorem 1.4 are true. Then, we have the following assertion:

> $(h(z); c_1, d_1), \ldots, (h(z); c_s, d_s)$ are just all finite slopes of tangential rays at P for right-branches centred at P.

Indeed, by statements (3, 5) in Theorem 1.4, there is at least one root α_k of $g(z)$ in $]c_k, d_k[$ such that $\pi(\alpha_k) = (h(z); c_k, d_k)$ for every $k \in \{1, \ldots, s\}$. By the definition of $\phi_+(\epsilon, z)$, we have $\phi_+(\epsilon, \alpha_k\epsilon) = 0$. Evidently, $\alpha_k\epsilon \in M$, $k = 1, \ldots, s$. According to Lemma 2.3(1), $(h(z); c_k, d_k)$ $(= \pi(\frac{\alpha_k\epsilon}{\epsilon}))$ is the slope of a tangential ray to some right-branch at P, $k = 1, \ldots, s$.

Conversely, assume that λ is the finite slope of a tangential ray to some right-branch at P. As in the proof of Lemma 2.3, let $\mathcal{R}_1, \ldots, \mathcal{R}_s$ be all distinct right-branches of $\mathcal{C}$ centred at P arranged in the order from below to above. Without loss of generality, we may assume that λ is the slope of the tangential ray to $\mathcal{R}_1$ at P. Thereby, the following sentence is valid in $\mathbb{R}$:

$$\forall d \left(d > 0 \longrightarrow \exists u \left(0 < u < \delta \wedge \forall (x, z_1, \ldots, z_s) \left(0 < x - a < u \wedge z_1 < \cdots < z_s \wedge \right.\right.\right.$$
$$\left.\left.\left. f(x, b + z_1) = \cdots = f(x, b + z_s) = 0 \to -d < \frac{z_1}{x - a} - \lambda < d \right)\right)\right).$$

By the Transfer Principle, the sentence above is also valid in R. Thereby, for the positive element ϵ in R, there is a $u_0 \in R$ such that $0 < u_0 < \delta$, and the following sentence is valid in R:

$$\forall(x, z_1, \ldots, z_s)\Big(0 < x - a < u_0 \wedge z_1 < \cdots < z_s \wedge$$
$$f(x, b + z_1) = \cdots = f(x, b + z_s) = 0 \to -\epsilon < \frac{z_1}{x - a} - \lambda < \epsilon\Big).$$

Put $\theta := u_0\epsilon$. Obviously $\theta \in M$, and $0 < \theta < u_0$. Likewise, there is an order-preserving $\mathbb{R}$-automorphism τ of R such that $\tau(\theta) = \epsilon$, since θ is positive and infinitesimal over $\mathbb{R}$. Let all roots of $f(a + \theta, y)$ in R be as follows: $y_1 < \cdots < y_s$. Observe that $z_1 < \cdots < z_s$, where $z_i := y_i - b$, $i = 1, \ldots, s$. By the validity of the second sentence, we have $-\epsilon < \frac{z_1}{\theta} - \lambda < \epsilon$. By the convexity of M in R, $\frac{z_1}{\theta} - \lambda \in M$. This yields $\frac{\tau(z_1)}{\epsilon} - \lambda = \tau(\frac{z_1}{\theta} - \lambda) \in \tau(M) = M$. Clearly, $\frac{\tau(z_1)}{\eta} \in A$, and $\pi\big(\frac{\tau(z_1)}{\eta}\big) = \lambda$.

On the other hand, by the equality $f(a + \theta, b + z_1) = 0$, we have $f\big(a + \epsilon, b + \tau(z_1)\big) = \tau\big(f(a + \theta, b + z_1)\big) = 0$, i.e., $g\big(\frac{\tau(z_1)}{\epsilon}\big) = 0$. Thus $\frac{\tau(z_1)}{\epsilon}$ is a bounded root of $g(z)$ in R. By statement (4) in Proposition 1.2, $\frac{\tau(z_1)}{\epsilon} \in]c_\ell, d_\ell[_R$ for some $\ell \in \{1, \ldots, s\}$. It follows that $\pi\big(\frac{\tau(z_1)}{\epsilon}\big) = (h(z); c_\ell, d_\ell)$ from statement (5) in Theorem 1.4. So we have $\lambda = (h(z); c_\ell, d_\ell)$. Hence, our assertion is verified, and the proof is completed. □

As for the computation of all infinite slopes of tangential rays to $\mathcal{C}$ at P, we may establish the following:

Lemma 2.6. *Let the notations be as above. Then we can effectively compute all infinite slopes of tangential rays to $\mathcal{C}$ at P.*

Proof. In what follows, we compute all infinite slopes of tangential rays at P only for right-branches centred at P.

Step 1. Compute the signed subresultant sequence of $\phi_+(\epsilon, z)$ relative to z, and delete those polynomials that are identically 0 from this sequence. Then a sequence of polynomials may be obtained as follows:

$$g_m, g_{m-1}, \ldots, g_0.$$

where $g_i \in F[\epsilon][z]$, $i = 0, \ldots, m$.

Step 2. Extract the trailing coefficients $u_i(z)$, $v_i(z)$ of g_i, $g_i(\epsilon z)$ as polynomials in ϵ respectively, $i = 0, \ldots, m$. Note: $u_i(z), v_i(z) \in F[z]$ for all i.

Step 3. Extract the trailing coefficient a_i of $u_i(z)$ and the leading coefficient b_i of $v_i(z)$, $i = 0, \ldots, m$.

Step 4. Compute respectively the modified numbers W_1, W_2, W_3 and W_4 of sign changes in the lists $\langle(-1)^{t_i}a_i \mid i = 0, \ldots, m\rangle$, $\langle(-1)^{\deg(v_i;z)}b_i \mid i = 0, \ldots, m\rangle$, $\langle b_i \mid i = 0, \ldots, m\rangle$ and $\langle a_i \mid i = 0, \ldots, m\rangle$, where t_i is the degree of the trailing term of $u_i(z)$, $i = 0, \ldots, m$.

Then it remains to prove the following claims:

(1) There are exactly $W_1 - W_2$ tangential rays to $\mathcal{C}$ at P having the slope $-\infty$.

(2) There are exactly $W_3 - W_4$ tangential rays to $\mathcal{C}$ at P having the slope ∞.

In the proof of Lemma 2.5, we have obtained a polynomial $h(z) \in F[z]$ and a finite number of open intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ in $\mathbb{R}$ such that all finite slopes of tangential rays at P for right-branches centred at P are just $(h(z); c_1, d_1)$, …, $(h(z); c_s, d_s)$. By Step 3, there exist two positive numbers d, D such that $d\langle\Delta, D\rangle \max\{|c_1|, |d_s|\}$, $\mathrm{sign}(u_i(-d)) = \mathrm{sign}((-1)^{t_i}a_i)$, $\mathrm{sign}(v_i(-D)) = \mathrm{sign}((-1)^{\deg(v_i;z)}b_i)$, $\mathrm{sign}(u_i(d)) = \mathrm{sign}(a_i)$, and $\mathrm{sign}(v_i(D)) = \mathrm{sign}(b_i)$,

$i = 0, \dots, m$. According to the definition of the ordering of R and the computation in Step 2, for $i = 0, \dots, m$, we further have

$$\begin{aligned}
&\operatorname{sign}(g_i(-d)) = \operatorname{sign}(u_i(-d)) = \operatorname{sign}((-1)^{t_i} a_i),\\
&\operatorname{sign}(g_i(-D\epsilon)) = \operatorname{sign}(v_i(-D)) = \operatorname{sign}((-1)^{\deg(v_i;z)} b_i),\\
&\operatorname{sign}(g_i(D\epsilon)) = \operatorname{sign}(v_i(D)) = \operatorname{sign}(b_i),\\
&\operatorname{sign}(g_i(d)) = \operatorname{sign}(u_i(d)) = \operatorname{sign}(a_i).
\end{aligned}$$

This implies that the modified numbers of sign changes in the lists $\langle g_i(-d) \mid i = 0, \dots, m\rangle$, $\langle g_i(-D\epsilon) \mid i = 0, \dots, m\rangle$, $\langle g_i(D\epsilon) \mid i = 0, \dots, m\rangle$ and $\langle g_i(d) \mid i = 0, \dots, m\rangle$ are W_1, W_2, W_3 and W_4 respectively. By Corollary 9.33 in Basu et al. (2003), the numbers of roots of $\phi_+(\epsilon, z)$ in the open intervals $]-d, -D\epsilon[_R$ and $]D\epsilon, d[_R$ are $W_1 - W_2$ and $W_3 - W_4$ respectively.

In order to verify the claims above, it suffices to prove that the number of tangential rays to $\mathcal{C}$ at P having the slope $-\infty$ is $W_1 - W_2$, and the number of tangential rays to $\mathcal{C}$ at P having the slope ∞ is $W_3 - W_4$.

Let $\mathcal{R}_1, \dots, \mathcal{R}_s$ be all distinct right-branches, arranged in the order from below to above, of $\mathcal{C}$ centred at P, and let $\eta_1 < \cdots < \eta_s$ be all roots of $\phi_+(\epsilon, z)$ in M. By Theorem 2.4, $\pi(\frac{\eta_i}{\epsilon})$ is the slope of the tangential rays to $\mathcal{R}_i$ at P, $i = 1, \dots, s$. Obviously, $\pi(\frac{\eta_1}{\epsilon}) \le \cdots \le \pi(\frac{\eta_s}{\epsilon})$. Assume that $\pi(\frac{\eta_1}{\epsilon}) = \cdots = \pi(\frac{\eta_\ell}{\epsilon}) = -\infty$ where $0 \le \ell \le s$, but $\pi(\frac{\eta_k}{\epsilon}) > -\infty$ whenever $k > \ell$. In this case, for $i \in \{1, \dots, \ell\}$, we have $\frac{\eta_i}{\epsilon} < -D$, because $\frac{\eta_i}{\epsilon}$ is a negative unbounded element. Thereby, $\eta_i < -D\epsilon$, $i = 1, \dots, \ell$. Moreover, for $i \in \{1, \dots, \ell\}$, we have $-d < \eta_i$, since $\eta_i \in M$ is infinitesimal over $\mathbb{R}$. This implies that $\eta_1 < \cdots < \eta_\ell$ are roots of $\phi_+(\epsilon, z)$ in $]-d, -D\epsilon[_R$.

Conversely, let η be any root of $\phi_+(\epsilon, z)$ in $]-d, -D\epsilon[_R$. Obviously $\eta \in A$. Suppose $\eta \notin M$. Then $\pi(\eta) \ne 0$. By the equality $f(a+\epsilon, b+\eta) = 0$, we have $f(a, b+\pi(\eta)) = \pi\big(f(a+\epsilon, b+\eta)\big) = 0$. From the inequality $-\Delta < -d < \eta < D\epsilon$, it follows that $-\Delta < -d = \pi(-d) < \pi(\eta) \le \pi(D\epsilon) = 0$. Thereby, the univariate polynomial $f(a, y)$ has two distinct roots $b, b+\pi(\eta)$ in the closed interval $[b - \Delta, b + \Delta]$; this contradicts condition (2.4) above. This yields $\eta \in M$. Hence, $\eta = \eta_p$ for some $p \in \{1, \dots, s\}$. Clearly, $\pi(\frac{\eta_p}{\epsilon}) \ne \infty$, since $\frac{\eta_p}{\epsilon} < 0$. Suppose $\pi(\frac{\eta_p}{\epsilon}) \ne -\infty$. By Lemma 2.3, $\pi(\frac{\eta_p}{\epsilon})$ is the finite slope of a tangential ray at P. According to the preceding argument, $\pi(\frac{\eta_p}{\epsilon}) = (h(z), c_q, d_q)$ for some $q \in \{1, \dots, m\}$. So we have $\pi(\frac{\eta_p}{\epsilon}) \in]c_q, d_q[\subseteq]c_1, d_s[$, and $|\pi(\frac{\eta_p}{\epsilon})| < max\{|c_1|, |d_s|\} < D$. On the other hand, $\pi(\frac{\eta_p}{\epsilon}) \le -D$, since $\frac{\eta_p}{\epsilon} < -D$. Then $|\pi(\frac{\eta_p}{\epsilon})| \ge D$, a contradiction. This implies $\pi(\frac{\eta_p}{\epsilon}) = -\infty$, and $\eta = \eta_p \in \{\eta_1, \dots, \eta_\ell\}$.

Hence, $\eta_1, \dots, \eta_\ell$ are just all roots of $\phi_+(\epsilon, z)$ in $]-d, -D\epsilon[_R$, and $\ell = W_1 - W_2$. This implies that claim (1) is valid. Likewise, claim (2) may be verified. This completes the proof. □

Remark. As is shown above, for a given polynomial $f(x, y) \in F[x, y]$, define such new polynomials $g_+(\epsilon, z)$, $g_-(\epsilon, z)$, $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ in $F[\epsilon][z]$ as follows:

$$\begin{aligned}
&g_+(\epsilon, z) := f(a+\epsilon, b+\epsilon z), \quad g_-(\epsilon, z) := f(a-\epsilon, b-\epsilon z),\\
&\phi_+(\epsilon, z) := f(a+\epsilon, b+z), \quad \text{and} \quad \phi_-(\epsilon, z) := f(a-\epsilon, b-z).
\end{aligned}$$

Clearly, we have

$$\phi_-(\epsilon, z) = \phi_+(-\epsilon, -z), \quad g_+(\epsilon, z) = \phi_+(\epsilon, \epsilon z), \quad \text{and} \quad g_-(\epsilon, z) = \phi_+(-\epsilon, -\epsilon z).$$

Then the following facts, which are important for the forthcoming algorithmic design, are obvious.

(1) For $\alpha, \beta \in R \cup \{-\infty, \infty\}$ with $\alpha < \beta$ and $\phi_+(\epsilon, \alpha)\phi_+(\epsilon, \beta) \neq 0$, the number of roots of $\phi_+(\epsilon, z)$ (or $\phi_-(\epsilon, z)$) in $]\alpha, \beta[_R$ is just that of roots of $f(a+\epsilon, y)$ (or $f(a-\epsilon, y)$) in $]b+\alpha, b+\beta[_R$ (or $]b-\beta, b-\alpha[_R$). Hence the following statement is true:

If $f_m, f_{m-1}, \ldots, f_0$ is the sequence obtained by deleting those polynomials that are identically 0 from the signed subresultant sequence of $f(x, y)$ relative to y, then the number of roots of $\phi_+(\epsilon, z)$ (or $\phi_-(\epsilon, z)$) in $]\alpha, \beta[_R$ is the difference between the modified numbers of $f_m(a+\epsilon, b+z), f_{m-1}(b+\epsilon, b+z), \ldots, f_0(a+\epsilon, b+z)$ at α and β (or $f_m(a-\epsilon, b-z)$, $f_{m-1}(a-\epsilon, b-z), \ldots, f_0(a-\epsilon, b-z)$ at β and α).

(2) If $h_+(z)$, $h_-(z)$ are the trailing coefficients of $g_+(\epsilon, z)$, $g_-(\epsilon, z)$ as polynomials in ϵ respectively, then $h_+(z) = \pm h_-(z)$.

(3) For an open interval $]a, b[$ in $\mathbb{R}$, α is a root of $g_+(\epsilon, z)$ (or $g_-(\epsilon, z)$) in $]a, b[_R$, if and only if $\alpha\epsilon$ is a root of $\phi_+(\epsilon, z)$ (or $\phi_-(\epsilon, z)$) in $]a\epsilon, b\epsilon[_R$.

On the basis of Lemmas 2.5 and 2.6, we can describe the following algorithm to compute the slopes of all tangential rays to $\mathcal{C}$ at P for a real plane algebraic curve $\mathcal{C}$ with a particular point P.

Algorithm 2.7 (*The Tangential Rays at a Particular Point on a Real Plane Algebraic Curve*).

Structure. a computable ordered subfield F of $\mathbb{R}$.

Input. a polynomial $f(x, y) \in F[x, y]$, which is primitive as a polynomial over $\mathbb{R}[x]$ in the variable y, $a, b \in F$ such that $f(a, b) = 0$.

Output. a polynomial $h(z) \in F[z]$, a set of isolating intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ for $h(z)$, two tuples $(q_{+0}, q_{+1}, \ldots, q_{+s}, q_{+(s+1)})$, $(q_{-0}, q_{-1}, \ldots, q_{-s}, q_{-(s+1)})$ of non-negative integers such that, for the curve $\mathcal{C}$ defined by the equation $f(x, y) = 0$ and its point $P(a, b)$, the following assertions are true:

- The numbers of right-branches and left-branches of $\mathcal{C}$ centred at P are $\sum_{k=0}^{s+1} q_{+k}$ and $\sum_{k=0}^{s+1} q_{-k}$ respectively.
- In the order from below to above, the respective slopes of the tangential rays to right-branches at P are as follows:

$$\overbrace{-\infty, \ldots, -\infty}^{q_{+0}}, \quad \overbrace{\alpha_1, \ldots, \alpha_1}^{q_{+1}}, \quad \ldots, \quad \overbrace{\alpha_s, \ldots, \alpha_s}^{q_{+s}}, \quad \overbrace{\infty, \ldots, \infty}^{q_{+(s+1)}}.$$

  where α_k is represented by $(h(z), c_k, d_k)$, $k = 1, \ldots, s$.
- In the order from above to below, the respective slopes of the tangential rays to left-branches at P are as follows:

$$\overbrace{-\infty, \ldots, -\infty}^{q_{-0}}, \quad \overbrace{\alpha_1, \ldots, \alpha_1}^{q_{-1}}, \quad \ldots, \quad \overbrace{\alpha_s, \ldots, \alpha_s}^{q_{-s}}, \quad \overbrace{\infty, \ldots, \infty}^{q_{-(s+1)}}.$$

  where α_k is represented by $(h(z), c_k, d_k)$, $k = 1, \ldots, s$.

Procedure.

(1) Compute the two new polynomials $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ in $F[\epsilon][z]$ defined as follows:

$$\phi_+(\epsilon, z) := f(a+\epsilon, b+z), \quad \phi_-(\epsilon, z) := f(a-\epsilon, b-z).$$

(2) Extract the trailing coefficient $h(z)$ of $\phi_+(\epsilon, \epsilon z)$ as a polynomial in ϵ. Moreover, by real root isolation for polynomials, compute a set of isolating intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ for $h(z)$.

(3) Compute the signed subresultant sequence of $f(x, y)$ relative to y, and delete those polynomials that are identically 0 from this sequence. A sequence may be obtained as follows:

$$f_m, f_{m-1}, \ldots, f_0,$$

where $f_i \in F[x, y], i = 0, \ldots, m$.

Thereby, the following sequences may be obtained as follows:

$$\phi_{+i}(\epsilon, z) := f_i(a + \epsilon, b + z), \quad i = 0, \ldots, m;$$
$$\phi_{-i}(\epsilon, z) := f_i(a - \epsilon, b - z), \quad i = 0, \ldots, m.$$

(4) Extract respectively the trailing coefficients $u_{+i}(z)$, $v_{+i}(z)$ of $\phi_{+i}(\epsilon, z)$, $\phi_{+i}(\epsilon, \epsilon z)$ as polynomials in ϵ, $i = 0, \ldots, m$.

Likewise, the trailing coefficients $u_{-i}(z)$, $v_{-i}(z)$ of $\phi_{-i}(\epsilon, z)$, $\phi_{-i}(\epsilon, \epsilon z)$ as polynomials in ϵ may be respectively extracted, $i = 0, \ldots, m$.

(5) Extract the trailing coefficient a_{+i} of $u_{+i}(z)$ and the leading coefficient b_{+i} of $v_{+i}(z)$ for $i = 0, \ldots, m$.

Likewise, extract the trailing coefficient a_{-i} of $u_{-i}(z)$ and the leading coefficient b_{-i} of $v_{-i}(z)$ for $i = 0, \ldots, m$.

(6) Count respectively the modified numbers W_{+1}, W_{+2}, W_{+3} and W_{+4} of sign changes in the lists $\langle(-1)^{t_{+i}} a_{+i} \mid i = 0, \ldots, m\rangle$, $\langle(-1)^{\deg(v_{+i};z)} b_{+i} \mid i = 0, \ldots, m\rangle$, $\langle b_{+i} \mid i = 0, \ldots, m\rangle$ and $\langle a_{+i} \mid i = 0, \ldots, m\rangle$, where t_{+i} is the degree of the trailing term of $u_{+i}(z)$, $i = 0, \ldots, m$.

Likewise, count respectively the modified numbers W_{-1}, W_{-2}, W_{-3} and W_{-4} of sign changes in the lists $\langle(-1)^{t_{-i}} a_{-i} \mid i = 0, \ldots, m\rangle$, $\langle(-1)^{\deg(v_{-i};z)} b_{-i} \mid i = 0, \ldots, m\rangle$, $\langle b_{-i} \mid i = 0, \ldots, m\rangle$ and $\langle a_{-i} \mid i = 0, \ldots, m\rangle$, where t_{-i} is the degree of the trailing term of $u_{-i}(z)$, $i = 0, \ldots, m$.

(7) For $k = 1, \ldots, s$, by Corollary 9.33 in Basu et al. (2003), count respectively the numbers q_{+k}, q_{-k} of roots of $\phi_+(\epsilon, z)$, $\phi_-(\epsilon, z)$ in $]c_k\epsilon, d_k\epsilon[$.

Put $q_{+0} := W_{+1} - W_{+2}$, $q_{-0} := W_{-2} - W_{-1}$, $q_{+(s+1)} := W_{+3} - W_{+4}$, and $q_{-(s+1)} := W_{-4} - W_{-3}$.

Proof of correctness. It follows from Lemmas 2.5 and 2.6 and their proofs. □

Complexity analysis. Now assume that the (total) degree of $f(x, y)$ is d.

(1) According to the complexity analysis of Algorithm 8.15 in Basu et al. (2003), computing $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ takes $O(\sum_{j=1}^{d} j(j + 1))$ arithmetic operations in F, i.e., the complexity of this computation is $O(d^3)$.

(2) Extracting the trailing coefficient $h(z)$ of $\phi_+(\epsilon, \epsilon z)$, which is considered as a polynomial in ϵ, is assumed to be costless. Observe that the degree of $h(z)$ is at most d. By the complexity analysis of Algorithm 10.41 in Basu et al. (2003), the complexity of computing a set of isolating intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ is $O((-\log_2(\lambda)+\ell+2)d^3)$, where ℓ is the minimal natural number such that $]-2^\ell, 2^\ell[$ contains all the real roots of $h(z)$, and λ the minimal distance between two complex roots of $h(z)$.

(3) According to the complexity analysis of Algorithm 8.73 in Basu et al. (2003), computing the sequence $f_m, f_{m-1}, \ldots, f_0$ requires $O(d^2)$ arithmetic operations in the domain $F[x]$. By Proposition 8.68 in Basu et al. (2003), the degrees of the polynomials in x produced in the intermediate computations are bounded by $2d^2$. The complexity of computing this sequence in F is $O(d^6)$. Observe that $m \le d$, and f_i is of degree $\le 2d(d - 1)$ for every $i \in \{0, 1, \ldots, m\}$. According to the complexity analysis of Algorithm 8.15 in Basu et al. (2003), computing the sequences $\phi_{+m}(\epsilon, z), \phi_{+(m-1)}(\epsilon, z), \ldots, \phi_{+0}(\epsilon, z)$ and $\phi_{-m}(\epsilon, z), \phi_{-(m-1)}(\epsilon, z), \ldots, \phi_{-0}(\epsilon, z)$

takes $O((d+1)\sum_{j=1}^{2d(d-1)} j(j+1))$ arithmetic operations in F, i.e., the complexity of this computation is $O(d^7)$.

(4) Extracting the trailing coefficients $u_{+i}(z)$, $v_{+i}(z)$ is assumed to be costless, $i = 0, \ldots, m$. Moreover, extracting the trailing coefficients $u_{-i}(z)$, $v_{-i}(z)$ is also assumed to be costless, $i = 0, \ldots, m$.

(5) As in (4), all the extractions in step (5) are assumed to be costless.

(6) Likewise, it is assumed to be costless to count the modified numbers W_{+1}, W_{+2}, W_{+3}, W_{+4}, W_{-1}, W_{-2}, W_{-3} and W_{-4}.

(7) Observe that $s \le d$, and the degrees of $\phi_{+i}(\epsilon, z)$ and $\phi_{-i}(\epsilon, z)$ are bounded by $O(d^2)$, $i = 0, \ldots, m$. By the complexity analysis of Algorithm 8.13 in Basu et al. (2003), the complexity of evaluating $\phi_{+i}(\epsilon, c_k\epsilon)$, $\phi_{+i}(\epsilon, d_k\epsilon)$, $\phi_{-i}(\epsilon, c_k\epsilon)$ and $\phi_{-i}(\epsilon, d_k\epsilon)$ is $O(d^2)$, $i = 0, \ldots, m$; $k = 1, \ldots, s$. Then the complexity of step (7) is $O(d^4)$.

Therefore, the complexity of Algorithm 2.7 is $O((-\log_2(\lambda) + \ell + 2)d^7)$. □

The effectiveness of Algorithm 2.7 may be assured, if all the coefficients of the polynomial $f(x, y)$ and the coordinates of the point P lie in $\mathbb{Q}$ or in $\mathbb{Q}(\beta)$ for some real algebraic number β.

Clearly, a more complicated problem is to find out all the tangential rays at each singular point for a real curve $\mathcal{C}$. The first job is to compute all the singular points of $\mathcal{C}$. Several ways of computing the singular points are presented in many references, e.g. Cellini et al. (1991), Gianni and Traverso (1983), Gonzalez-Vega and El Kahoui (1996), Keyser et al. (2000), Roy and Szpirglas (1990), Sakkalis (1991), and Seidel and Wolper (2005). In what follows, we shall use the so-called univariate representations to code the singular points.

Let $\mathcal{C}$ be a real curve defined by the equation $f(x, y) = 0$, where $f(x, y) \in \mathbb{Q}[x, y]$. Without loss of generality, we may assume that the polynomial $f(x, y)$ is squarefree. In this case, the ideal of $\mathbb{R}[x, y]$ generated by f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is zero-dimensional. With the aid of Theorem 8.81 in Becker et al. (1993), we may obtain a triple (η, μ, ξ) with $\eta, \mu, \xi \in \mathbb{Q}[x]$ satisfying the following conditions: (i) η is squarefree; (ii) $\deg(\eta) > \max\{\deg(\mu), \deg(\xi)\}$; (iii) $\{(\mu(a), \xi(a)) \mid a \in \mathbb{R}(\sqrt{-1}), \text{ and } \eta(a) = 0\}$ is the set of all zeros of the system $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ in $\mathbb{R}(\sqrt{-1})^2$; and (iv) $\{(\mu(a), \xi(a)) \mid a \in \mathbb{R}, \text{ and } \eta(a) = 0\}$ is the set of all zeros of the system $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ in $\mathbb{R}^2$. If $]\sigma_1, \tau_1[, \ldots,]\sigma_r, \tau_r[$ is a set of isolating intervals for $\eta(x)$, then all real roots of $\eta(x)$ may be coded by (η, σ_k, τ_k), $k = 1, \ldots, r$. Hence, all real zeros of the system $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, i.e., the coordinates of all singular points of $\mathcal{C}$, are $(\mu(\beta_k), \xi(\beta_k))$, where $\beta_k = (\eta, \sigma_k, \tau_k)$, $k = 1, \ldots, r$. For this reason, such a triple (η, μ, ξ) is called *a univariate representation* for all singular points of $\mathcal{C}$. In what follows, write $((\eta, \sigma_k, \tau_k), \mu, \xi)$ for the singular point with coordinates $(\mu(\beta_k), \xi(\beta_k))$, $k = 1, \ldots, r$. Thereby, the tangential rays at each particular singular point $((\eta, \sigma_k, \tau_k), \mu, \xi)$ can be determined by applying Algorithm 2.7 to the case in which $F = \mathbb{Q}(\beta)$ and $(a, b) = (\mu(\beta_k), \xi(\beta_k))$.

In view of Algorithm 2.7, we are able to describe the following algorithm, and leave its proof to the reader as an exercise.

Algorithm 2.8 (*The Tangential Rays at Each Singular Point on a Real Plane Algebraic Curve*).

Structure. the field $\mathbb{Q}$ of rational numbers.

Input. a polynomial $f(x, y) \in \mathbb{Q}[x, y]$, which is primitive as a polynomial over $\mathbb{Q}[x]$ in the variable y, a univariate representation (η, μ, ξ) for all singular points of the curve $\mathcal{C}$ defined by the equation $f(x, y) = 0$.

Output. all encodings of the singular points of $\mathcal{C}$ as follows: $((\eta, \sigma_k, \tau_k), \mu, \xi)$, $k = 1, \ldots, r$, a polynomial $h(z) \in \mathbb{Q}[z]$, a set of isolating intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ for $h(z)$. For every

$k = 1, \ldots, r$, two tuples $(q_{+k0}, q_{+k1}, \ldots, q_{+ks}, q_{+k(s+1)})$, $(q_{-k0}, q_{-k1}, \ldots, q_{-ks}, q_{-k(s+1)})$ of non-negative integers such that, for every $k = 1, \ldots, r$, the following assertions are true:

- The numbers of right-branches and left-branches of $\mathcal{C}$ centred at $((\eta, \sigma_k, \tau_k), \mu, \xi)$ are respectively $\sum_{j=0}^{s+1} q_{+kj}$ and $\sum_{j=0}^{s+1} q_{-kj}$.
- In the order from below to above, the respective slopes of the tangential rays to right-branches at $((\eta, \sigma_k, \tau_k), \mu, \xi)$ are as follows:

$$\overbrace{-\infty, \ldots, -\infty}^{q_{+k0}}, \quad \overbrace{\alpha_1, \ldots, \alpha_1}^{q_{+k1}}, \quad \ldots, \quad \overbrace{\alpha_s, \ldots, \alpha_s}^{q_{+ks}}, \quad \overbrace{\infty, \ldots, \infty}^{q_{+k(s+1)}}.$$

  where $\alpha_j = (h(z), c_j, d_j)$, $j = 1, \ldots, s$.
- In the order from above to below, the respective slopes of the tangential rays to left-branches at $((\eta, \sigma_k, \tau_k), \mu, \xi)$ are as follows:

$$\overbrace{-\infty, \ldots, -\infty}^{q_{-k0}}, \quad \overbrace{\alpha_1, \ldots, \alpha_1}^{q_{-k1}}, \quad \ldots, \quad \overbrace{\alpha_s, \ldots, \alpha_s}^{q_{-ks}}, \quad \overbrace{\infty, \ldots, \infty}^{q_{-k(s+1)}}.$$

  where $\alpha_j = (h(z), c_j, d_j)$, $j = 1, \ldots, s$.

Procedure.

(1) Compute a set of isolating intervals $]\sigma_1, \tau_1[, \ldots,]\sigma_r, \tau_r[$ for $\eta(x)$.

(2) Compute the resultant $\mathrm{Res}(\eta, f(\mu + \epsilon, \xi + \epsilon z); x)$ of η and $f(\mu + \epsilon, \xi + \epsilon z)$ relative to x, and extract the trailing coefficient $h(z)$ of $\mathrm{Res}(\eta, f(\mu + \epsilon, \xi + \epsilon z); x)$ as a polynomial over $\mathbb{Q}[z]$ in ϵ.

(3) Compute a set of isolating intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ for $h(z)$.

(4) Compute the signed subresultant sequence of $f(x, y)$ relative to y, and delete those polynomials that are identically 0 from this sequence. Such a sequence is obtained as follows:

$$f_m, f_{m-1}, \ldots, f_0,$$

where $f_i \in F[x, y]$, $i = 0, \ldots, m$.

Consequently, the following sequences may be obtained as follows:

$$g_{+i}(\epsilon, x, z) = f_i(\mu + \epsilon, \xi + z), \quad i = 0, \ldots, m.$$
$$g_{-i}(\epsilon, x, z) = (-1)^i f_i(\mu - \epsilon, \xi - z), \quad i = 0, \ldots, m.$$

(5) For every $k = 1, \ldots, r$, denote $\beta_k = (\eta, \sigma_k, \tau_k)$. Extract respectively the trailing coefficients $u_{+ki}(\beta_k, z)$, $v_{+ki}(\beta_k, z)$ of $g_{+i}(\epsilon, \beta_k, z)$, $g_{+i}(\epsilon, \beta_k, \epsilon z)$ as polynomials in ϵ, $i = 0, \ldots, m$.

Likewise, extract respectively the trailing coefficients $u_{-ki}(\beta_k, z)$, $v_{-ki}(\beta_k, z)$ of $\phi_{-i}(\epsilon, \beta_k, z)$, $\phi_{-i}(\epsilon, \beta_k, \epsilon z)$ as polynomials in ϵ, $i = 0, \ldots, m$.

(6) For every pair (k, i) with $1 \le k \le r$ and $0 \le i \le m$, determine the sign e_{+ki} of the trailing coefficient of $u_{+ki}(\beta_k, z)$ and the sign e'_{+ki} of the leading coefficient of $v_{+ki}(\beta_k, z)$.

Likewise, determine the sign e_{-ki} of the trailing coefficient of $u_{-ki}(\beta_k, z)$ and the sign e'_{-ki} of the leading coefficient of $v_{-ki}(\beta_k, z)$.

(7) For every $k = 1, \ldots, r$, count respectively the modified numbers V_{+k1}, V_{+k2}, V_{+k3} and V_{+k4} of sign changes in the lists $\langle (-1)^{t_{+ki}} e_{+ki} \mid i = 0, \ldots, m \rangle$, $\langle (-1)^{\deg(v_{+ki}; z)} e'_{+ki} \mid i = 0, \ldots, m \rangle$, $\langle e'_{+ki} \mid i = 0, \ldots, m \rangle$ and $\langle e_{+ki} \mid i = 0, \ldots, m \rangle$, where t_{+ki} is the degree of the trailing term of $u_{+ki}(\beta_k, z)$, $i = 0, \ldots, m$.

Likewise, count respectively the modified numbers V_{-k1}, V_{-k2}, V_{-k3} and V_{-4k} of sign changes in the lists $\langle (-1)^{t_{-ki}} e_{-ki} \mid i = 0, \ldots, m\rangle$, $\langle (-1)^{\deg(v_{-ki};z)} e'_{-ki} \mid i = 0, \ldots, m\rangle$, $\langle e'_{-ki} \mid i = 0, \ldots, m\rangle$ and $\langle e_{-ki} \mid i = 0, \ldots, m\rangle$, where t_{-ki} is the degree of the trailing term of $u_{-ki}(\beta_k, z)$, $i = 0, \ldots, m$.

(8) For every pair (k, j) with $1 \le k \le r$ and $1 \le j \le s$, count respectively the numbers q_{+kj}, q_{-kj} of roots of $f(\mu(\beta_k) + \epsilon, \xi(\beta_k) + \epsilon z)$, $f(\mu(\beta_k) - \epsilon, \xi(\beta_k) - \epsilon z)$ in $]c_j, d_j[_R$.

(9) For every $k = 1, \ldots, r$, compute the integers q_{+k0}, q_{-k0}, $q_{+k(s+1)}$ and $q_{-k(s+1)}$ such that $q_{+k0} := V_{+k1} - V_{+k2}$, $q_{-k0} := V_{-k2} - V_{-k1}$, $q_{+k(s+1)} := V_{+k3} - V_{+k4}$, and $q_{-k(s+1)} := V_{-k4} - V_{-k3}$. $\square$

Moreover, we can establish an interesting result on the number of tangential rays with the same slope. In what follows, for the sake of convenience, a tangential ray with slope $-\infty$ or ∞ is said to have the slope $\pm\infty$. This implies that all the tangential rays with slope $\pm\infty$ at a given point $P(a, b)$ possess the same tangent $x - a = 0$.

First, we establish the following lemma.

Lemma 2.9. *Let $\langle a_i \mid i = 1, \ldots, s\rangle$, $\langle b_i \mid i = 1, \ldots, s\rangle$ be two lists of non-zero elements in R, and $e_i \in \{-1, 1\}$, $i = 1, \ldots, s$. If V_1, V_2, V_3 and V_4 are the numbers of sign changes in the lists $\langle a_i \mid i = 1, \ldots, s\rangle$, $\langle b_i \mid i = 1, \ldots, s\rangle$, $\langle e_i a_i \mid i = 1, \ldots, s\rangle$ and $\langle e_i b_i \mid i = 1, \ldots, s\rangle$ respectively, then $V_1 - V_2$ has the same parity as $V_3 - V_4$.*

Proof. By the definition of sign changes, we have

$$V_1 = \sum_{i=1}^{s-1} \frac{1}{2}(1 - \mathrm{sign}_R(a_i a_{i+1})),$$

$$V_2 = \sum_{i=1}^{s-1} \frac{1}{2}(1 - \mathrm{sign}_R(b_i b_{i+1})),$$

$$V_3 = \sum_{i=1}^{s-1} \frac{1}{2}(1 - \mathrm{sign}_R(e_i e_{i+1} a_i a_{i+1})),$$

$$V_4 = \sum_{i=1}^{s-1} \frac{1}{2}(1 - \mathrm{sign}_R(e_i e_{i+1} b_i b_{i+1})).$$

Then we have

$$\begin{aligned} V_3 - V_4 &= \sum_{i=1}^{s-1} \frac{1}{2}(\mathrm{sign}_R(e_i e_{i+1} b_i b_{i+1}) - \mathrm{sign}_R(e_i e_{i+1} a_i a_{i+1})), \\ &= \sum_{i=1}^{s-1} \frac{1}{2} e_i e_{i+1}(\mathrm{sign}_R(b_i b_{i+1}) - \mathrm{sign}_R(a_i a_{i+1})). \end{aligned}$$

Obviously, the following congruences hold:

$$e_i e_{i+1} \equiv 1 \pmod 2, \quad i = 1, \ldots, s-1.$$

Since $\frac{1}{2}(\mathrm{sign}_R(b_i b_{i+1}) - \mathrm{sign}_R(a_i a_{i+1}))$ is an integer for $i = 1, \ldots, s-1$, we have

$$\frac{1}{2}e_ie_{i+1}(\text{sign}_R(b_ib_{i+1}) - \text{sign}_R(a_ia_{i+1})) \equiv \frac{1}{2}(\text{sign}_R(b_ib_{i+1}) - \text{sign}_R(a_ia_{i+1})) \pmod 2,$$

where $i = 1, \ldots, s-1$.

So we have $V_3 - V_4 \equiv \sum_{i=1}^{s-1} \frac{1}{2}(\text{sign}_R(b_ib_{i+1}) - \text{sign}_R(b_ib_{i+1})) \pmod 2$, i.e., $V_3 - V_4 \equiv V_1 - V_2 \pmod 2$. This completes the proof. □

Corollary. *Let $\langle a_i \mid i = 1, \ldots, s\rangle$, $\langle b_i \mid i = 1, \ldots, s\rangle$ be two lists of elements in R with $a_1b_1 \neq 0$, and $e_i \in \{-1, 1\}$, $i = 1, \ldots, s$. If W_1, W_2, W_3 and W_4 are the modified numbers of sign changes in the lists $\langle a_i \mid i = 1, \ldots, s\rangle$, $\langle b_i \mid i = 1, \ldots, s\rangle$, $\langle e_ia_i \mid i = 1, \ldots, s\rangle$ and $\langle e_ib_i \mid i = 1, \ldots, s\rangle$ respectively, then $W_1 - W_2$ has the same parity as $W_3 - W_4$.*

Proof. It follows immediately from Lemma 2.9 and the definition of the modified numbers of sign changes. □

Theorem 2.10. *Let the notations be as above, and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$. Then the number of tangential rays with slope λ of $\mathcal{C}$ at P is even.*

Proof. Write n for the number of tangential rays with slope λ of $\mathcal{C}$ at P. Without loss of generality, we may assume $n \neq 0$. Now consider the two possible cases as follows:

Case 1. $\lambda = \pm\infty$. In the following consideration, we adopt the same symbols as in the procedure of Algorithm 2.7. By Algorithm 2.7, we have

$$\begin{aligned} n &= (W_{+1} - W_{+2}) + (W_{+3} - W_{+4}) + (W_{-2} - W_{-1}) + (W_{-4} - W_{-3}) \\ &= (W_{+1} - W_{+4}) - (W_{-1} - W_{-4}) + (W_{+3} - W_{+2}) - (W_{-3} - W_{-2}). \end{aligned}$$

Observe that $\phi_{-i}(\epsilon, z) = \phi_{+i}(-\epsilon, -z)$ for $i = 0, \ldots, m$. So we have $t_{-i} = t_{+i}$, $\deg(v_{+i}; z) = \deg(v_{-i}; z)$, $a_{-i} = e_ia_{+i}$, and $b_{-i} = e_i'b_{+i}$, where $e_i, e_i' \in \{-1, 1\}$, $i = 0, \ldots, m$. By the corollary of Lemma 2.9, both $(W_{+1} - W_{+4}) - (W_{-1} - W_{-4})$ and $(W_{+3} - W_{+2}) - (W_{-3} - W_{-2})$ are even. Hence n is even.

Case 2. $\lambda \in \mathbb{R}$. In this case, by a suitable rotation of axes, all the tangential rays with slope λ may be converted into the tangential rays with slope $\pm\infty$. By the argument in Case 1, n must be even. The proof is completed. □

As an immediate consequence of Theorem 2.10, we have the following result, which may be found as Theorem 9.5.7 in Bochnak et al. (1998).

Corollary. *Let the notations be as above. Then the number of left-branches plus the number of right-branches of $\mathcal{C}$ centred at P is even.*

Actually, Theorem 2.10 and its corollary are not used in our tangent algorithm, because they are only two qualitative results on the tangential rays and the half-branches of a real plane algebraic curve.

3. Examples

In the final section, we will treat some examples with the aid of the computer algebraic system **Maple**. For details of **Maple**, refer to Heck (1993).

Example 1 (*Reconsideration of the Example in the Introduction*). Investigate all tangents of the real curve $\mathcal{C}$ at the origin, where $\mathcal{C}$ is defined by $2x^5 - x^4y + xy^2 - y^3 = 0$.

According to Algorithm 2.7, we implement the computations as follows:

(1) In this case, we have

$$\phi_+(\epsilon, z) = 2\epsilon^5 - \epsilon^4 z + \epsilon z^2 - z^3,$$
$$\phi_-(\epsilon, z) = -2\epsilon^5 + \epsilon^4 z - \epsilon z^2 + z^3.$$

(2) Extract the trailing coefficients $h(z)$ of $\phi_+(\epsilon, \epsilon z)$ as a polynomial in ϵ as follows:

$$h(z) = -z^3 + z^2.$$

By real root isolation for polynomials, we obtain a set of isolating intervals $]-1, \frac{1}{2}[$, $]\frac{1}{2}, 2[$ for $h(z)$ such that $]-1, \frac{1}{2}[$ contains only the root 0 of $h(z)$, and $]\frac{1}{2}, 2[$ contains only the root 1 of $h(z)$.

(3) The signed subresultant sequence of $f(x, y)$ relative to y is computed as follows:

$$f_3 = 2x^5 - x^4 y + xy^2 - y^3, \quad f_2 = -x^4 + 2xy - 3y^2,$$
$$f_1 = -17x^5 + (6x^4 - 2x^2)y, \quad f_0 = 4x^{12} + 71x^{10} + 8x^8.$$

Consequently, the two sequences may be obtained as follows:

$$\phi_{+3}(\epsilon, z) = 2\epsilon^5 - \epsilon^4 z + \epsilon z^2 - z^3, \quad \phi_{+2}(\epsilon, z) = -\epsilon^4 + 2\epsilon z - 3z^2,$$
$$\phi_{+1}(\epsilon, z) = -17\epsilon^5 + (6\epsilon^4 - 2\epsilon^2)z, \quad \phi_{+0}(\epsilon, z) = 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8;$$
$$\phi_{-3}(\epsilon, z) = -2\epsilon^5 + \epsilon^4 z - \epsilon z^2 + z^3, \quad \phi_{-2}(\epsilon, z) = -\epsilon^4 + 2\epsilon z - 3z^2,$$
$$\phi_{-1}(\epsilon, z) = 17\epsilon^5 - (6\epsilon^4 - 2\epsilon^2)z, \quad \phi_{-0}(\epsilon, z) = 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8.$$

(4) For $i = 0, \dots, 3$, extracting respectively the trailing coefficients $u_{+i}(z)$, $v_{+i}(z)$ of $\phi_{+i}(\epsilon, z)$, $\phi_{+i}(\epsilon, \epsilon z)$ as polynomials in ϵ, we have

$$u_{+3}(z) = -z^3, \quad u_{+2}(z) = -3z^2, \quad u_{+1}(z) = -2z, \quad u_{+0}(z) = 8;$$
$$v_{+3}(z) = z^2 - z^3, \quad v_{+2}(z) = 2z - 3z^2, \quad v_{+1}(z) = -2z, \quad v_{+3}(z) = 8.$$

For $i = 0, \dots, 3$, extracting respectively the trailing coefficients $u_{-i}(z)$, $v_{-i}(z)$ of $\phi_{-i}(\epsilon, z)$, $\phi_{-i}(\epsilon, \epsilon z)$ as polynomials in ϵ, we have

$$u_{-3}(z) = z^3, \quad u_{-2}(z) = -3z^2, \quad u_{-1}(z) = 2z, \quad u_{-3}(z) = 8;$$
$$v_{-3}(z) = -z^2 + z^3, \quad v_{-2}(z) = 2z - 3z^2, \quad v_{-1}(z) = 2z, \quad v_{-0}(z) = 8.$$

(5) Extracting the trailing coefficient a_{+i} of $u_{+i}(z)$ and the leading coefficient b_{+i} of $v_{+i}(z)$ for $i = 0, \dots, 3$, we have

$$a_{+3} = -1, a_{+2} = -3, a_{+1} = -2, a_{+3} = 8; \quad b_{+3} = -1, b_{+2} = -3, b_{+1} = -2, b_{+0} = 8.$$

Likewise, extracting the trailing coefficient a_{-i} of $u_{-i}(z)$ and the leading coefficient b_{-i} of $v_{-i}(z)$ for $i = 0, \dots, 3$, we have

$$a_{-3} = 1, a_{-2} = -3, a_{-1} = 2, a_{-0} = 8; \quad b_{-3} = 1, b_{-2} = -3, b_{-1} = 2, b_{-0} = 8.$$

(6) Counting the modified numbers V_{+1}, V_{+2}, V_{+3} and V_{+4} of sign changes in the lists $\langle(-1)^3 a_{+3}, (-1)^2 a_{+2}, -a_{+1}, a_{+0}\rangle$, $\langle(-1)^3 b_{+3}, (-1)^2 b_{+2}, -b_{+1}, b_{+0}\rangle$, $\langle a_{+3}, a_{+2}, a_{+1}, a_{+0}\rangle$ and $\langle b_{+3}, b_{+2}, b_{+1}, b_{+0}\rangle$, we have

$$V_{+1} = 2, \quad V_{+2} = 2, \quad V_{+3} = 1, \quad V_{+4} = 1.$$

Likewise, the numbers V_{-1}, V_{-2}, V_{-3} and V_{-4} of sign changes in the lists $\langle(-1)^3a_{-3}, (-1)^2a_{-2}, -a_{-1}, a_{-0}\rangle$, $\langle(-1)^3b_{-3}, (-1)^2b_{-2}, -b_{-1}, b_{-0}\rangle$, $\langle a_{-3}, a_{-2}, a_{-1}, a_{-0}\rangle$ and $\langle b_{-3}, b_{-2}, b_{-1}, b_{-0}\rangle$ may be counted as follows:

$$V_{-1} = 1, \quad V_{-2} = 1, \quad V_{-3} = 2, \quad V_{-4} = 2.$$

Thereby, $V_{+1} - V_{+2} = V_{-2} - V_{-1} = V_{+3} - V_{+4} = V_{-4} - V_{-3} = 0$.

(7) By the substitutions $z = -\epsilon$, $\frac{1}{2}\epsilon$ and 2ϵ, we have

$$\begin{aligned}
&\langle \phi_{+3}(\epsilon, -\epsilon), \phi_{+2}(\epsilon, -\epsilon), \phi_{+1}(\epsilon, -\epsilon), \phi_{+0}(\epsilon, -\epsilon)\rangle \\
&\quad = \langle 3\epsilon^5 + 2\epsilon^3, -\epsilon^4 - 5\epsilon^2, -23\epsilon^5 + \epsilon^3, 4\epsilon^{12} + 71\epsilon^{12} + 8\epsilon^8\rangle, \\
&\left\langle \phi_{+3}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{+2}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{+1}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{+0}\left(\epsilon, \frac{1}{2}\epsilon\right)\right\rangle \\
&\quad = \left\langle \frac{3}{2}\epsilon^5 + \frac{1}{8}\epsilon^3, -\epsilon^4 + \frac{1}{4}\epsilon^2, -14\epsilon^5 - \epsilon^3, 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8\right\rangle, \\
&\langle \phi_{+3}(\epsilon, 2\epsilon), \phi_{+2}(\epsilon, 2\epsilon), \phi_{+1}(\epsilon, 2\epsilon), \phi_{+0}(\epsilon, 2\epsilon)\rangle \\
&\quad = \langle -4\epsilon^3, -\epsilon^4 - 8\epsilon^2, -5\epsilon^5 - 4\epsilon^3, 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8\rangle; \\
&\langle \phi_{-3}(\epsilon, -\epsilon), \phi_{-2}(\epsilon, -\epsilon), \phi_{-1}(\epsilon, -\epsilon), \phi_{-0}(\epsilon, -\epsilon)\rangle \\
&\quad = \langle -3\epsilon^5 - 2\epsilon^3, -\epsilon^4 - 5\epsilon^2, 23\epsilon^5 + 2\epsilon^3, 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8\rangle, \\
&\left\langle \phi_{-3}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{-2}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{-1}\left(\epsilon, \frac{1}{2}\epsilon\right), \phi_{-0}\left(\epsilon, \frac{1}{2}\epsilon\right)\right\rangle \\
&\quad = \left\langle -\frac{3}{2}\epsilon^5 - \frac{1}{8}\epsilon^3, -\epsilon^4 + \frac{1}{4}\epsilon^2, 14\epsilon^5 + \epsilon^3, 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8\right\rangle, \\
&\langle \phi_{-3}(\epsilon, 2\epsilon), \phi_{-2}(\epsilon, 2\epsilon), \phi_{-1}(\epsilon, 2\epsilon), \phi_{-0}(\epsilon, 2\epsilon)\rangle \\
&\quad = \langle 4\epsilon^3, -\epsilon^4 - 8\epsilon^2, 5\epsilon^5 + 4\epsilon^3, 4\epsilon^{12} + 71\epsilon^{10} + 8\epsilon^8\rangle.
\end{aligned}$$

Clearly, these lists have the respective modified numbers of sign changes as follows:

$$2, 2, 1; \quad 1, 1, 2.$$

By Corollary 9.33 in Basu et al. (2003), both $\phi_+(\epsilon, z)$ and $\phi_-(\epsilon, z)$ have no root in $]-\epsilon, \frac{1}{2}\epsilon[$, but the numbers of roots of $\phi_+(\epsilon, z)$, $\phi_-(\epsilon, z)$ in $]\frac{1}{2}\epsilon, 2\epsilon[$ are both 1.

According to Algorithm 2.7, the numbers of right-branches and left-branches of $\mathcal{C}$ centred at the origin are both 1, and the slopes of the tangential rays to the right-branch and the left-branch at the origin are both 1. This implies that the line $x - y = 0$ is actually the only tangent to $\mathcal{C}$ at the origin. The diagram of the curve $\mathcal{C}$ is shown in Fig. 1.

With the aid of the computer algebra system **Maple**, Algorithm 2.7 has been made into a general program to determinate the tangential rays at a given point on a real plane algebraic curve defined by a polynomial equation with rational coefficients.

The following example was done on a Pentium IV computer with 128 MB RAM. For the given curve in this example, its topology was determined in Gonzalez-Vega and Necula (2002).

Example 2. Let $\mathcal{C}$ be a curve defined by the equation $f(x, y) = 0$, where $f(x, y) = x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8 - 4x^6 - 12x^4y^2 - 12x^2y^4 - 4y^6 + 16x^2y^2$. Determine all the tangential rays to the curve $\mathcal{C}$ at the origin.

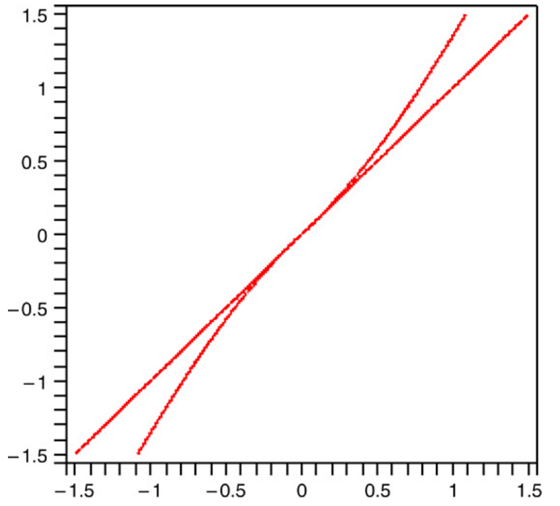

Fig. 1.

At the cost of CPU time 0.08 s, the following results appeared on the screen:

From below to above, the right slopes at (0, 0) are as follows:

$[-\infty, 1], [(z, -1, 1), 2], [\infty, 1]$.

From above to below, the left slopes at (0, 0) are as follows:

$[-\infty, 1], [(z, -1, 1), 2], [\infty, 1]$.

In the above, the phrase "the right(left) slopes" means "the slopes of the tangential rays to the right(left)-branches".

This implies that the number of half-branches centred at the origin is 8, and that the (distinct) tangents to $\mathcal{C}$ at the origin are just the lines $x = 0$ and $y = 0$. The diagram of the curve $\mathcal{C}$ is shown in Fig. 2.

Now we proceed to determine all the tangential rays at each singular point by applying Algorithm 2.8.

Example 3. Let $\mathcal{C}$ be a real curve defined by the equation $f(x, y) = 0$, where $f(x, y) = x^4 + 2x^2y^2 - x^2 + y^4 - y^2 + 2 - 2xy - 2y + 2x$. Find out all the tangential rays at each singular point of $\mathcal{C}$.

With the aid of Gröbner bases, we obtain such a univariate representation (η, μ, ξ) for all singular points of $\mathcal{C}$, where $\eta = x^2 + x$, $\mu = x$, and $\xi = x + 1$.

According to Algorithm 2.8, we implement the computations as follows:

(1) By real root isolation for polynomials, we obtain a set of isolating intervals $]-\frac{3}{2}, -\frac{1}{2}[$, $]-\frac{1}{2}, 1[$ for η such that $]-\frac{3}{2}, -\frac{1}{2}[$ contains only the root -1 of η, and $]-\frac{1}{2}, 1[$ contains only the root 0 of η.

(2) Compute the resultant of η and $f(\mu + \epsilon, \xi + \epsilon z)$ relative to x as follows:

$$\begin{aligned}
&\mathrm{Res}(\eta, f(\mu + \epsilon, \xi + \epsilon z); x)\\
&\quad = (z^8 + 4z^6 + 6z^4 + 4z^2 + 1)\epsilon^8\\
&\qquad + (4z^7 - 4z^6 + 12z^5 - 12z^4 + 12z^3 - 12z^2 + 4z - 4)\epsilon^7
\end{aligned}$$

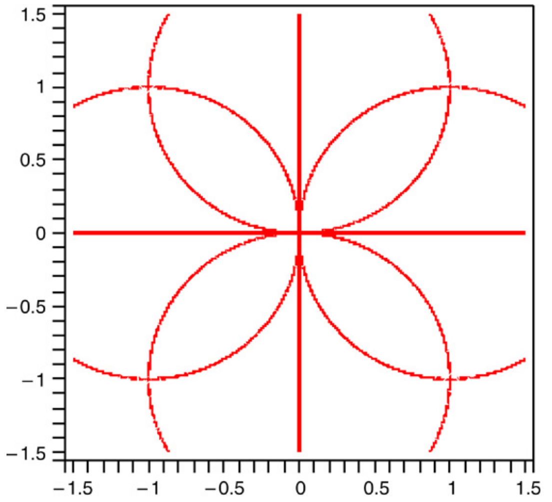

Fig. 2.

$$+ (6z^6 - 20z^5 + 18z^4 - 40z^3 + 18z^2 - 20z + 6)\epsilon^6$$
$$+ (4z^5 - 28z^4 + 32z^3 - 32z^2 + 28z - 4)\epsilon^5$$
$$+ (5z^4 - 12z^3 + 30z^2 - 12z + 5)\epsilon^4.$$

Moreover, the trailing coefficient $h(z)$ of $\mathrm{Res}(\eta, f(\mu+\epsilon, \xi+\epsilon z); x)$ as a polynomial over $\mathbb{Q}[z]$ in ϵ is extracted as follows:

$$h(z) = 5z^4 - 12z^3 + 30z^2 - 12z + 5.$$

(3) By real root isolation, it is verified that $h(z)$ has no real root.

According to Algorithm 2.8, there is no tangential ray at each singular point of $\mathcal{C}$. Hence, every singular point of $\mathcal{C}$ is isolated. Actually, $\mathcal{C}$ consists of two isolated points $(-1, 0)$, $(0, 1)$, since $f(x, y) = (x - y + 1)^2 + (x^2 + y^2 - 1)^2$.

Algorithm 2.8 has been made into a general program to determinate the tangential rays at all the singular points for a real plane algebraic curve. For Example 3, our program yields the result as shown as above at the cost of CPU time 0.25 s.

Example 4. Let $\mathcal{C}$ be a real curve defined by the equation $f(x, y) = 0$, where $f(x, y) = x^2y^3 - x^4 + yx^2 + 2y^5 - 2y^2x^2 + y^3 + x^2 - y$. Find out all the tangential rays at each singular point of $\mathcal{C}$.

With the aid of the computation of Gröbner bases, a univariate representation (η, μ, ξ) for all singular points of $\mathcal{C}$ was obtained, where $\eta = x^6+x^4+15x^2-9$, $\mu = x$, and $\xi = \frac{1}{12}(x^4+4x^2+3)$.

At the cost of CPU time 4.85 s, the following results appeared on the screen:

$$h = 399\,424z^{12} - 20\,464z^{10} - 311\,960z^8 - 80\,937z^6 + 57\,240z^4 + 32\,400z^2 + 5184.$$

From below to above, the right slopes at $((\eta, -2, 0), \mu, \xi)$ are as follows:

$$\left[\left(h, -\frac{31}{16}, -\frac{27}{32}\right), 1\right], \left[\left(h, 0, \frac{27}{32}\right), 1\right].$$

From above to below, the left slopes at $((\eta, -2, 0), \mu, \xi)$ are as follows:

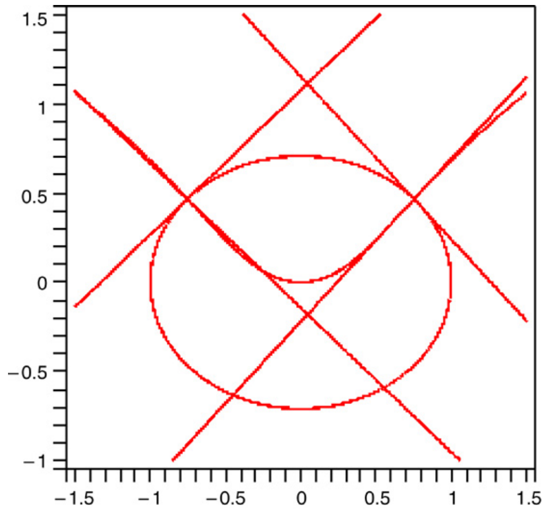

Fig. 3.

$$\left[\left(h, -\frac{31}{16}, -\frac{27}{32}\right), 1\right], \left[\left(h, 0, \frac{27}{32}\right), 1\right].$$

From below to above, the right slopes at $((\eta, 0, 2), \mu, \xi)$ are as follows:

$$\left[\left(h, -\frac{27}{32}, 0\right), 1\right], \left[\left(h, \frac{27}{32}, \frac{31}{16}\right), 1\right].$$

From above to below, the left slopes at $((\eta, 0, 2), \mu, \xi)$ are as follows:

$$\left[\left(h, -\frac{27}{32}, 0\right), 1\right], \left[\left(h, \frac{27}{32}, \frac{31}{16}\right), 1\right].$$

This implies that the singular points of the curve $\mathcal{C}$ are just $((\eta, -2, 0), \mu, \xi)$, $((\eta, 0, 2), \mu, \xi)$, and their tangential rays are respectively described as above. The diagram of the curve $\mathcal{C}$ is shown in Fig. 3.

Finally, we consider the following example, which was explicitly investigated in section 6 of Cucker et al. (1989). In Cucker et al. (1989), the authors used the rational Puiseux expansions and the Thom's codification of real algebraic numbers to compute the local and global analytic branches of a real algebraic curve. In view of the following computation, it would seem that our algorithm is faster for the problem of computing the real tangents.

Example 5. Let $\mathcal{C}$ be a real curve defined by the equation $f(x, y) = 0$, where $f(x, y) = y^3 - (x+1)y^2 + (x^2-1)y - x^3 - x^2 + x + 1$. Find out all the tangential rays at each singular point of $\mathcal{C}$.

With the aid of the computation of Gröbner bases, such a univariate representation $(x^2 + x, x, x+1)$ for all singular points of $\mathcal{C}$ was obtained. Hence, all the singular points of $\mathcal{C}$ are $(0, 1)$ and $(-1, 0)$. At the cost of CPU time 0.035 s, the following results appeared on the screen:

From below to above, the right slopes at $(0, 1)$ are as follows:

$$\left[\left(z^2 - z, -1, \frac{1}{2}\right), 1\right], \left[\left(z^2 - z, \frac{1}{2}, 2\right), 1\right].$$

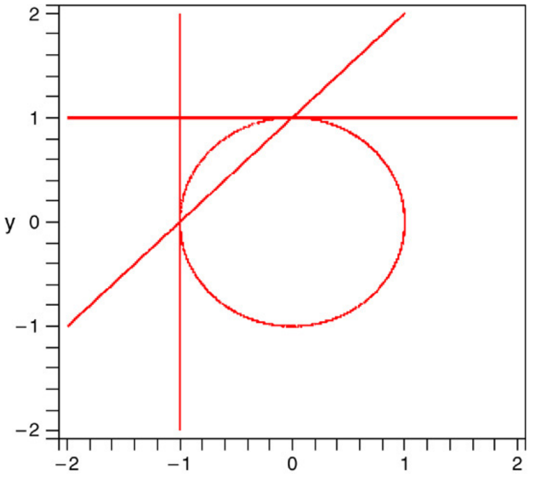

Fig. 4.

From above to below, the left slopes at (0, 1) are as follows:

$$\left[\left(z^2 - z, -1, \frac{1}{2}\right), 1\right], \left[\left(z^2 - z, \frac{1}{2}, 2\right), 1\right].$$

From below to above, the right slopes at $(-1, 0)$ are as follows:

$[-\infty, 1], [(-z + 1, 0, 2), 1], [\infty, 1]$.

From above to below, the left slopes at $(-1, 0)$ are as follows:

$[(-z + 1, 0, 2), 1]$.

This result implies that the numbers of right and left half-branches centred at (0, 1) are both 2, but the numbers of right and left half-branches centred at $(-1, 0)$ are 3 and 1 respectively. Moreover, the required tangential rays are respectively described as above. The diagram of $\mathcal{C}$ is shown in Fig. 4.

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References

Arnon, D.S., Collins, G.E., McCallum, S., 1984. Cylindrical algebraic decomposition II: An adjacent algorithm for the plane. SIAM J. Comput. 13, 878–889.

Bajaj, G.L., Hoffman, C.M., Lynch, R.E., Hopcroft, J.E.H., 1988. Tracing surface intersections. Comput. Aided Geom. Design 5, 285–307.

Basu, S., Pollack, R., Roy, M.-F., 2003. Algorithms in Real Algebraic Geometry. In: Algorithms and Computation in Math., vol. 10. Springer-Verlag, Berlin.

Becker, T., Weispfenning, V., Kredel, H., 1993. Gröbner Bases: A Computational Approach to Commutative Algebra. Springer-Verlag, New York, Berlin, Heidelberg.

Bochnak, J., Coste, M., Roy, M.-F., 1998. Real Algebraic Geometry. Springer-Verlag, New York, Berlin, Heidelberg.

Cellini, P., Gianni, P., Traverso, C., 1991. Algorithms for the shape of semialgebraic sets: A new approach. In: Lecture Notes in Computer Science, vol. 539. Springer–Verlag, Berlin, Heidelberg, pp. 1–18.

Cucker, F., Gonzalez-Vega, L., Rosello, F., 1991. On algorithms for real algebraic curves. In: Effective Methods in Algebraic Geometry. In: Progress in Math, vol. 94. Birkhausser, pp. 63–88.

Cucker, F., Pardo, L.M., Raimondo, M., Recio, T., Roy, M.F., 1989. On the Computation of the Local and Global Analytic Branches of a Real Algebraic Curve. In: Proceedings of the 5th International Conference AAECC-5. In: Lecture Notes in Computer Science, vol. 356. Springer-Verlag, Berlin, Heidelberg, pp. 161–181.

Gianni, P., Traverso, C., 1983. Shape determination of real curves and surfaces. Ann. Univ. Ferrara Sez VII Sec. Math. XXIX, 87–109.

Gonzalez-Vega, L., El Kahoui, M., 1996. An improved upper complexity bound for the topology computation of a real algebraic plane curve. J. Complexity 12, 527–544.

Gonzalez-Vega, L., Necula, I., 2002. Efficient topology determination of implicitly defined algebraic plane curves. Comput. Aided Geom. Design 19, 719–743.

Gonzalez-Vega, L., Recio, T., Lombardi, H., Roy, M.-F., 1998. Sturm-Habicht sequences, determinants and real roots of univariate polynomials. In: Quantifier Elimination and Cylindrical Algebraic Decomposition (Linz, 1993). In: Texts Monogr. Symbol. Comput. Springer-Verlag, Vien, New York, pp. 300–316.

Heck, A., 1993. Introduction to Maple. Springer-Verlag, New York, Berlin, Heidelberg.

Hoffman, C.M., 1988. Algebraic curves. In: Rice, J. (Ed.), Mathematical Aspects of Scientific Software. In: IMA Volumes in Math. and Appl., vol. 14. Springer-Verlag, Berlin, pp. 101–122.

Hong, H., 1996. An effective method for analyzing the topology of plane real algebraic curves. Math. Comput. Simulation 42, 572–582.

Keyser, J., Culver, T., Manocha, D., Krishnan, S., 2000. Efficient and exact manipulation of algebraic points and curves. Comput. Aided Design 32 (11), 649–662.

Knebusch, M., 1973. On the extension of real places. Comment. Math. Helv. 48, 354–369.

Lam, T.Y., 1980. The Theory of Ordered Fields. In: Lecture Notes in Pure and Appl. Math., vol. 55. M. Dekker, New York.

Loos, R., 1982. Computing in Algebraic Extensions. In: Computing Supplementum, vol. 4. Springer-Verlag, Vienna, pp. 173–187.

Prestel, A., 1984. Lectures on Formally Real Fields. In: Lecture Notes in Math., vol. 1093. Springer-Verlag, Berlin, Heidelberg, New York.

Roy, M,-F., Szpirglas, A., 1990. Complexity of the computation of cylindrical decomposition and topology of real algebraic curves using Thom's Lemma. In: Lecture Notes in Math., vol. 1420. Springer-Verlag, Berlin, Heidelberg, New York, pp. 223–236.

Sakkalis, T., 1991. The topological configuration of a real algebraic curve. Bull. Austral. Math. Soc. 43 (1), 37–50.

Sakkalis, T., Farouki, R., 1990. Singular points of algebraic curves. J. Symbolic Comput. 9, 405–421.

Seidel, R., Wolper, N., 2005. On the Exact computation of the topology of real algebraic curves (Exploiting a little more Geometry and a little less Algebra). In: 21st Annual ACM Symposium on Computational Geometry, pp. 107–15.

Walker, R.J., 1978. Algebraic Curves. Springer-Verlag, New York, Heidelberg, Berlin.